Bootstrap Methods for Univariate and Multivariate Volatility

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To my family and friends

Abstract

This thesis focuses on developing bootstrap procedures for realized volatility estimators, which are often used to measure the financial market volatility based on high frequency intraday data.
Unlike the commonly used continuous-time stochastic volatility (SV) models, discrete-time models for the logarithmic returns with underlying time varying volatility functions are investigated. In these models, the innovation term is not necessarily normally distributed and weak dependence is allowed. To describe this weak dependence, we make use of the geometric-moment contracting (GMC) property as an underlying assumption. The central limit results for our discrete-time models are given, which in some cases differ from the results for SV models.
For the univariate discrete-time models, we propose a kernel estimator to capture the underlying spot volatility structure, and thereafter estimate the underlying innovations. In chapter 2, the innovations are assumed to be independent. We propose a nonparametric i.i.d. bootstrap procedure by resampling the estimated noise innovations, and a nonparametric wild bootstrap procedure by generating pseudo-noise that imitates correctly the first and second order properties of the underlying noise. Because of the simple (independent) setup of this model, we are able to give results for the realized bipower variation, which is a more general volatility estimator than the realized volatility. In chapter 3, the innovation term is assumed to be a time series with weak dependence. We combine the kernel volatility estimation with the linear process bootstrap of McMurry and Politis (2010). This proposal highly depends on the accuracy of the kernel estimation. For our proposed kernel estimator, the application is restricted to those time series, in which the autocovariance decays geometrically.
In the multivariate discrete-time models, the varying volatility structure cannot be estimated anymore. However, the underlying volatility is assumed to be a smooth function, and the return process is therefore locally stationary. Based on this property, we propose to use the local bootstrap approach of Shi (1991) for the model with independent innovations, and the local block bootstrap of Paparoditis and Politis

(2002) for the model with weak dependence. As an alternative to the local block bootstrap method, we propose a local dependent wild bootstrap procedure by application of the dependent wild bootstrap of Shao (2010) in nonoverlapping local windows. The validity of all bootstrap proposals is proved, and the finite sample properties of some proposals are investigated in simulation studies.

Zusammenfassung

Der Schwerpunkt dieser Arbeit liegt auf der Entwicklung von Bootstrap-Verfahren für auf hochfrequenten Daten basierende Schätzung von Volatilitäten im Finanzmarkt.
Im Gegensatz zu den üblicherweise verwendeten zeitstetigen stochastischen Volatilitätsmodellen (SV), untersuchen wir zeitdiskrete Modelle für logarithmierte Renditen mit zugrunde liegenden variierenden Volatilitätsfunktionen. Bei diesen Modellen ist der Innovationsterm nicht zwingend normalverteilt, zudem wird schwache Abhängigkeit zugelassen, welche über die Geometric-Moment-Contracting-Eigenschaft beschrieben wird. Die asymptotischen Verteilungen der Volatilitätsschätzer sind für die zeitdiskreten Modelle unter unterschiedlichen Bedingungen angegeben. In einigen Fällen unterscheiden sich die Resultate zu denen für die SV-Modelle.
Für die univariaten zeitdiskreten Modelle benutzen wir einen Kern-Schätzer, um die zugrunde liegende variierende Volatilität zu erfassen und darauf basierend die Innovationen zu schätzen. In Kapitel 2 wird die Unabhängigkeitsbedingung für den Innovationsterm angenommen. Wir schlagen ein nichtparametrisches Verfahren vor, bei dem der IID-Bootstrap-Ansatz auf die berechneten Innovationen angewendet wird. Als eine Alternative wird ein nichtparametrisches Wild-Bootstrap-Verfahren vorgeschlagen. Dabei werden Pseudo-Zufallszahlen erzeugt, deren erstes und zweites Moment mit denen der Innovationen übereinstimmen. In Kapitel 3 ist der Innovationsterm eine schwach abhängige Zeitreihe. Wir kombinieren das Kernschäzungsverfahren mit dem Linear-Process-Bootstrap von McMurry und Politis (2010). Dieser Vorschlag hängt stark von der Genauigkeit der Kernschätzung ab. Die Anwendung ist auf Zeitreihen mit geometrisch abfallender Autokovarianzfunktion begrenzt.
Bei mehrdimensionaler Modellierung kann die variierende Volatilität nicht mehr geschätzt werden. Jedoch wird die Volatilität mit einer stetig differenzierbaren Funktion modelliert. Der Rendite-Prozess kann als eine lokal stationäre Zeitreihe betrachtet werden. Basierend auf dieser Eigenschaft schlagen wir vor, das Local-Bootstrap-Verfahren von Shi (1991) für das unabhängige Modell und das Local-Block-Bootstrap-Verfahren von Paparoditis and Politis (2002) für das schwach abhängige Modell

einzusetzen. Im Vergleich zum Local-Block-Bootstrap wird ein Local-Dependent-Wild-Bootstrap entwickelt, bei dem der Dependent-Wild-Bootstrap von Shao (2010) in nicht überlappenden lokalen Zeitintervallen eingesetzt wird. Die Validität aller Bootstrap-Ansätze wird bewiesen und ihre Eigenschaft bei begrenztem Stichprobeumfang wird durch Simulationen untersucht.

Contents

1 Introduction

A time series is a sequence of observations, which are usually arranged to their recording time. This type of data are collected in many areas, for instance, daily exchange rate and daily share price in finance; daily temperature and rainfall in meteorology, monthly data for import and export in economics, etc. One of the main reasons for recording and analyzing time series data is that people hope to understand the generating mechanism of the data, and then to predict a future value. Typically, it is assumed that the observations $X_1, \ldots, X_n$ are generated by an unknown stochastic process $(X_t)_{t\in\mathbb{Z}}$ on a probability space $(\Omega, \mathcal{F}, P)$. When the characteristic properties of this underlying process $(X_t)_{t\in\mathbb{Z}}$ are to be estimated, *time series analysis* comes into play.

In this section we shall first give a short overview over definitions and results in time series analysis that are related to our results. For a more detailed exposition on these topics see e.g. Brockwell and Davis (1991) or Kreiss and Neuhaus (2006). After that, we introduce a series of widely used estimators for the financial market volatility and the asymptotic distributions of them under a general stochastic volatility model. A short overview of bootstrap methods, which is an alternative tool to the classical statistical analysis, is then given, and in the end the main results of this thesis are outlined.

1.1 Preliminary definitions in time series analysis

Stationary time series

The most essential assumption in time series analysis is stationarity, which describes the invariance of some statistical properties under time shift. Specifically, a time series $(X_t)_{t\in\mathbb{Z}}$ is said to be **strictly stationary** or **strongly stationary**, if the joint distribution of $(X_{t_1}, \ldots, X_{t_i})$ is identical to that of $(X_{t_1+h}, \ldots, X_{t_i+h})$ for all sets of time indices $t_1, \ldots, t_i$ and integers h. This is a strong assumption. To define a weaker version of stationarity, we need the co-called autocovariance function. Let $(X_t)_{t\in\mathbb{Z}}$ be a real-valued time series with $EX_t^2 < \infty$ for all $t \in \mathbb{Z}$. The **autocovariance function** of $(X_t)_{t\in\mathbb{Z}}$ is defined as

$$\gamma_X(t, t+h) := Cov\left(X_t, X_{t+h}\right) = E\left[\left(X_t - EX_t\right)\left(X_{t+h} - EX_{t+h}\right)\right],$$

for all $t, h \in \mathbb{Z}$.
A real-valued time series $(X_t)_{t\in\mathbb{Z}}$ is said to be **(weakly) stationary**, if for all $t \in \mathbb{Z}$, $EX_t^2 < \infty$, $EX_t = \mu$, which is a constant and $\gamma_X(t, t+h)$ only depends on the so-called lag h.
From the definitions, it is easy to find that if $(X_t)_{t\in\mathbb{Z}}$ is strictly stationary with finite second-order moment, then $(X_t)_{t\in\mathbb{Z}}$ is weakly stationary as well. The converse is not true in general. In this thesis, stationarity refers to weak stationarity, as it is often done in the literature. And in the context of stationarity, the notation $\gamma_X(h)$ is often used instead of $\gamma_X(t, t+h)$, as it is independent of the time point t.

Using autocovariance is one common way to describe the inner dependence structure of a time series. A natural estimator of it is the so-called sample autocovariance function. Let the observations $X_1, \ldots, X_n$ of a stationary time series $(X_t)_{t\in\mathbb{Z}}$ be given. $\overline{X} := \frac{1}{n}\sum_{t=1}^{n} X_t$ denotes the sample mean. The **sample autocovariance function** is defined as

$$\widehat{\gamma}_X(h) := \frac{1}{n}\sum_{t=1}^{n-|h|}\left(X_t - \overline{X}\right)\left(X_{t+|h|} - \overline{X}\right), \quad |h| < n.$$

We restrict ourselves to real-valued time series for notational reasons and introduce several stationary time series models that are related to our results in the following.

- **White noise:** A time series $(\varepsilon_t)_{t\in\mathbb{Z}}$ is said to be a white noise, if $E\varepsilon_t = 0$, $E\varepsilon_t^2 = \sigma^2 \in (0, \infty)$ for all $t \in \mathbb{Z}$ and $Cov(\varepsilon_s, \varepsilon_t) = 0$ for all $s \neq t$. From the definition, it is easy to check that white noise is weakly stationary. With an additional assumption, that $(\varepsilon_t)_{t\in\mathbb{Z}}$ is independent and identically distributed (i.i.d.), a white noise process is even strictly stationary. White noise is one of the most simple time series, which is often used in modeling uncorrelated data and in constructing some other time series.

- **MA Process:** The moving average process $(X_t)_{t\in\mathbb{Z}}$ of order q ($MA(q)$) is defined as

$$X_t = \varepsilon_t + \theta_1\varepsilon_{t-1} + \ldots + \theta_q\varepsilon_{t-q}, \quad t \in \mathbb{Z},$$

where $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a white noise process, $\theta_1,\ldots,\theta_q$ are real-valued coefficients with $\theta_q \neq 0$ and $q \in \mathbb{N}$. As a sum of uncorrelated stationary processes, $MA(q)$ is stationary.

- **Linear process:** A time series $(X_t)_{t\in\mathbb{Z}}$ is said to be a linear process, if it has the representation

$$X_t = \mu + \sum_{i=-\infty}^{\infty} a_i\varepsilon_{t-i}, \quad t \in \mathbb{Z},$$

where $(\varepsilon_t)_{t\in\mathbb{Z}}$ is i.i.d. white noise and $\sum\limits_{i=\infty}^{\infty} |a_i| < \infty$. A linear process is always strictly stationary.
A time series $(X_t)_{t\in\mathbb{Z}}$ is said to be **causal** in the sense that it only depends on the 'history' not on the 'future'. In our case, if $a_i = 0$ for all $i < 0$, we have a causal linear process, which is also an $MA(\infty)$ representation.

- **AR Process:** An autoregressive process $(X_t)_{t\in\mathbb{Z}}$ of order p ($AR(p)$) is defined as

$$X_t = \varepsilon_t + \phi_1 X_{t-1} + \ldots + \phi_q X_{t-p}, \quad t \in \mathbb{Z},$$

where $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a white noise process, $\phi_1,\ldots,\phi_q$ are real-valued coefficients with $\phi_p \neq 0$ and $p \in \mathbb{N}$.
If all the roots of the equation $1 - \sum\limits_{k=1}^{p} \phi_k z^k = 0$ lie outside the unit circle, there exists a causal linear representation with real coefficients, and in this case the AR process is stationary.

- **ARMA process:** An autoregressive moving average process of orders (p,q) (short $ARMA(p,q)$) is defined as

$$X_t - \sum_{i=1}^{p} \phi_i X_{t-i} = \sum_{j=1}^{q} \theta_j \varepsilon_{t-j} + \varepsilon_t, \quad t \in \mathbb{Z},$$

where $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a white noise process, $\phi_1,\ldots,\phi_p$ and $\theta_1,\ldots,\theta_q$ are real-valued coefficients with $\phi_p \neq 0$, $\theta_q \neq 0$ and $p,q \in \mathbb{N}$. If $\phi_i = 0$ for all $i \in \{1,\ldots,p\}$, we have an $MA(q)$ process, while if $\theta_j = 0$ for all $j \in \{1,\ldots,q\}$, we have an $AR(p)$ process. Under the same assumption as for AR processes, a causal linear representation exists and an ARMA process is then stationary.

There are some more widely used nonlinear processes, such as ARCH, GARCH, etc. For details see e.g. Brockwell and Davis (1991) or Kreiss and Neuhaus (2006).

Local stationary time series

In the recent years nonstationary time series are more and more investigated. It is a more difficult situation because of the time varying properties (parameters, moments, etc.). One of the mostly discussed nonstationary processes is the class of so-called locally stationary processes, which were introduced by Dahlhaus (1997). As example, the so-called linear locally stationary process is defined as

$$X_t = \mu(t) + \sum_{i=-\infty}^{\infty} a_i(t)\varepsilon_{t-i}, \quad , t \in \mathbb{Z},$$

where $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a white noise process, $\mu(t)$ is a drift function, and the time varying coefficient functions $a_j(t)$ need to fulfill certain assumptions, which originate from nonparametric statistics.

Loosely speaking, this type of process is close to a stationary process in each local time interval. There are some more local stationary time series, i.e. time varying autoregressive processes, time varying ARCH-models, time varing GARCH-models, etc. For an overview over locally stationary processes, see e.g. Dahlhaus (2012).

Weak dependence

To quantify inner dependence of a time series, a dependence measure, named physical or functional dependence measure was originally proposed by Wu (2005). Upon this dependence measure several asymptotic theories are given, see for example Shao and Wu (2007), Wu (2011) and Wu and Zhou (2011) among others.

Let $\varepsilon_i, i \in \mathbb{Z}$, be i.i.d. random variables and H be a measurable function. We have a stationary process of the form

$$X_i = H\left(\ldots, \varepsilon_{i-1}, \varepsilon_i\right).$$

Let $(\varepsilon'_t)_{t\in\mathbb{Z}}$ be an i.i.d. copy of $(\varepsilon_t)_{t\in\mathbb{Z}}$, and X'_i be a coupled version of X_i with ε_0 being replaced by ε'_0:

$$X'_i = H(\ldots, \varepsilon_{-1}, \varepsilon'_0, \varepsilon_1, \ldots, \varepsilon_i).$$

For $p > 0$ and a random variable X, if $\|X\|_p := (E|X|^p)^{\frac{1}{p}} < \infty$, we say $X \in \mathcal{L}^p$. Let $X \in \mathcal{L}^p$, $p > 0$. The **physical dependence measure** is defined as

$$\delta_p(i) := \|X_i - X'_i\|_p.$$

Furthermore, the geometric-moment contracting (GMC) condition will be adopted to describe the dependence property of a stationary process.

Let $\widetilde{X}_i = H\left(\dots, \varepsilon'_{-1}, \varepsilon'_0, \varepsilon_1, \dots, \varepsilon_i\right)$ be another coupled version of X_i. We say that the process $(X_i)_{i\in\mathbb{Z}}$ satisfies $GMC(p)$, $p > 0$, if there exist $C > 0$ and $\rho \in (0,1)$, such that for all $n \in \mathbb{N}$,

$$E\left(|\widetilde{X}_i - X_i|^p\right) \leq C\rho^i$$

The GMC property indicates that the dependence between X_i and $(\dots, \varepsilon_{-1}, \varepsilon_0)$ reduces geometrically quickly. It was mentioned by Wu (2011), that the $GMC(p)$ property is equivalent to the physical dependence measure $\delta_p(n) = \mathcal{O}\left(\rho^n\right)$ for some $\rho \in (0,1)$.

Many time series models satisfy GMC under suitable conditions, for instance:

- **Linear Processes:** Let (a_i) be real coefficients of a causal linear process. It holds $\delta_p(i) = c_0|a_i|$, where $c_0 = \|\varepsilon_0 - \varepsilon'_0\|_p < \infty$. If $|a_i| = \mathcal{O}\left(r^i\right)$ for some $r \in (0,1)$, the causal linear process satisfies $GMC(p)$. We have in this case $\sum_{i=0}^{\infty} |a_i| < \infty$, which leads to the absolute summability of autocovariance.

- **ARMA(p,q):** Let $\phi_1, \dots, \phi_p$ and $\theta_1, \dots, \theta_q$ be real coefficients of a ARMA(p,q) process. If all the roots of the equation $1 - \sum_{k=1}^{p} \phi_k z^k = 0$ lie outside the unit circle, there exists a causal linear representation with real coefficients (b_i), where $|b_i| = \mathcal{O}\left(r^i\right)$ for some $r \in (0,1)$. It leads to the GMC property.

Furthermore, under certain conditions on the coefficients, ARCH and GARCH models satisfies GMC. For details see e.g. Wu and Min (2005). Sufficient conditions for GMC under general nonlinear autoregressive models are given by Shao and Wu (2007), and several specific nonlinear models are investigated there.
We use the GMC property as an underlying assumption to describe weak dependence for our asymptotic theory.

1.2 Financial market volatility estimation

Volatility describes the fluctuation of financial instruments, can be understood as a measure of risk, therefore plays an essential role in financial markets. Unlike some other market variables, volatility is not directly observable. Based on so-called high frequency financial data, a range of model-free volatility estimators, which measure the ex post variation of asset prices, were given and extensively analyzed in the last twenty years. High-frequency financial data are observations on financial instruments, such as price and return, taken at a fine time scale, i.e. at 1 minute frequency. There are even tick by tick ultra-high-frequency data. The advancement and integration of computer technology resulted in a increasing availability of this type of data in recent years. Based on the following standard continues time model for the log-price process, known as stochastic volatility (SV) model, some estimators this type, named realized measures, will be introduced in this section.

Let the log-return process be given by

$$dP_t = \mu_t dt + \sigma_t dW_t,$$

where μ_t denotes the drift, σ_t is a volatility term, and W_t is a standard Brownian motion. We consider a fixed time interval $[0, 1]$, which means a day. For a partition over a day, $0 = t_0 < t_1 < \ldots < t_n = 1$, the intraday log-returns are defined as

$$X_{i,n} := P_{t_i} - P_{t_{i-1}}, \qquad i = 1, \ldots, n.$$

Under this framework the **integrated volatility (IV)**(also named integrated variance) over a day is defined as

$$IV := \int_0^1 \sigma_t^2 dt,$$

which is an important value to quantify the variation of the price process.
The so-called quadratic variation (QV) of a price process over a day is given by

$$QV = \text{p-lim}_{n\to\infty} \sum_{i=1}^{n} X_{i,n}^2,$$

for any sequence of partitions $0 = t_0 < t_1 < \ldots < t_n = 1$ with $\sup\limits_{1\le i\le n} |t_i - t_{i-1}| \to 0$ as $n \to \infty$. p-lim denotes the probability limit. In independent work by Barndorff-Nielsen and Shephard (2001) and Andersen and Bollerslev (1998) appeared that, under certain assumptions, the quadratic variation is equal to integrated volatility for the above SV model.
Based on this result, a simple but most commonly used estimator of IV, known as **realized volatility (RV)** was proposed, which is defined as

$$RV := \sum_{i=1}^{n} X_{i,n}^2,$$

where n is the number of intraday observations. The consistency of RV under certain conditions was first noted in Andersen and Bollerslev (1998). For its application in some empirical work in the early time, see e.g. Andersen, Bollerslev, Diebold and Ebens (2001) and Andersen, Bollerslev, Diebold and Labys (2001). An asymptotic approximation to the distribution of realized volatility was given by Barndorff-Nielsen and Shephard (2002). See also Barndorff-Nielsen and Shephard (2004c) for an analysis of the finite sample behavior of the distribution of RV via Monte Carlo simulation. Integrated covariance matrix is the multivariate version of IV. Based on multivariate high-frequency data, Barndorff-Nielsen and Shephard (2004b) have proposed an asymptotic distribution theory for realized covariance, which is the multivariate version of RV.

A generalized estimator, known as **realized power variation (RPV)** is defined as

$$RPV(r) := \sum_{i=1}^{n} |X_{i,n}|^r, \quad r \geq 0.$$

Barndorff-Nielsen, Graversen and Shephard (2004) gives a review of some work on this estimator . When $r = 2$, we have exactly the usual realized volatility.
A more general consistent estimator, the co-called **realized bipower variation (RBP)**, introduced by Barndorff-Nielsen and Shephard (2004a) is defined as

$$RBV(r,s) := n^{\frac{r+s}{2}-1} \sum_{i=2}^{n} |X_{i,n}|^r |X_{i-1,n}|^s, \quad r, s \geq 0.$$

In this article the asymptotic behaviors of RPV and RBP for the stochastic volatility model with jump component were investigated. The robustness of RPV for $r \in (0,2)$ and RBP for $\max\{r,s\} < 2$ to finite activity jumps was shown. $RBV(1,1)$ is one of the often applied realized bipower variation estimators for IV, which is robust to rare jumps. A Central Limit Theorem for RPV and RBV was given by Barndoff-Nielsen et al. (2006).
MinRV and MedRv are two more jump robust estimators, which are extended from the bipower and tripower variation (see e.g. Andersen et al. (2012)).

It is worth mentioning that all jump robust measures above do not estimate exactly IV, but IV multiplied by a coefficient, which depends on some moment of W_t. In the stochastic volatility model we know that W_t is a standard Brownian motion. This coefficient can be therefore computed and then we have the estimated IV. In other words, without knowing these moments of W_t, we couldn't estimate IV via these jump robust measures.

There are other challenges by the volatility estimation besides jumps. One of them is the market microstructure noise effect: Bid-ask, rounding, etc could lead to errors in the price observations. The realized volatility is biased when microstructure noise is present and this bias will be worse, if data with higher frequency is used. Another challenge is the so-called Epps effect, which is caused by non-synchronous price observations. To deal with microstructure noise effect, several consistent estimators of IV were proposed and investigated. In the univariate context, see, for example, Zhang et al. (2005), Christensen and Podolskij (2007) and Barndorff-Nielsen et al. (2008). Zhang (2011) investigated the combination of microstructure noise and Epps effect. In the multivariate context, Barndorff-Nielsen et al. (2011) provide a multivariate realised kernel estimator, which is consistent for non-synchronous data with microstructure noise.

In this thesis, we focus on realized volatility and realized bipower variation in the univariate case and realized covariance in the multivariate case, which are simple to compute and still effective in the low-high-frequency area. We considered a discrete-time model with equidistant intraday data, without market microstructure noise nor jumps.

1.3 Bootstrap methods

In time series analysis, sample data is typically assumed to be generated by a random mechanism. Analyzing data is aimed to get knowing of the underlying 'randomness'. Based on a set of observations, people might be able to estimate some statistical quantities, such as using the realized volatility to estimate the integrated volatility as already mentioned above. But how accurate is the estimator? And what is the distribution of the statistics of interest involving the estimator? To deal with these questions, a usual way is to approximate the unknown true distributions of the statistics with the help of asymptotic distributions obtained from Central Limit Theorems. Based on the limiting normal distributions, confidence intervals of statistics of interest can be given. This approach, named *normal approximation*, has some substantial drawbacks. The limiting normal distribution is an asymptotic result (as sample size goes to infinity). The distribution based on a finite sample might differ form the limiting normal distribution, so that the approximation error might be

large. Additionally, the finite sample distribution might be heavily skewed, which cannot be approximated by a normal distribution because of its symmetry. Furthermore, to obtain a confidence interval via normal approximation, people need the asymptotic variance for which a consistent estimator might not be available.
Over the last decades, *resampling* methods are introduced into time series analysis and highly developed. One of the most famous and widely used methods is the so-called *bootstrap*, which was primarily introduced by Efron (1979) for i.i.d. observations. Assume, a sample set $\{X_1, \cdots, X_n\}$ of i.i.d. random variables is given. The distribution of a statistic $T_n = T_n(X_1, \ldots, X_n)$ is to be approximated. The i.i.d. bootstrap procedure is as follows:

- **Step 1:** Draw randomly with replacement from the original sample n times to get bootstrap sample $\{X_1^*, \ldots, X_n^*\}$.
- **Step 2:** Compute the corresponding bootstrap statistic $T^* = T_n(X_1^*, \ldots, X_n^*)$.
- **Step 3:** Repeat the last two steps N times to get N values of T^*. Use the empirical distribution of these N values to approximate the desired distribution.

With the help of bootstrap methods, people might be able to improve the approximation of the desired distribution (compared to the normal approximation), especially to show the skewness of the finite sample distribution, if it exists. In general situation, people need to prove the validity of a bootstrap procedure by showing that T and T^* have the same limiting distribution. However, proving validity does not give us any information about the quality of the bootstrap approximation. Monte Carlo simulation is a helpful technique to present it.

To construct a consistent bootstrap procedure does not mean that the whole underlying data generating process has to be mimicked. For instance, if the asymptotic distribution depends only on the autocovariance, the correct imitating of the second-order properties of the underlying process might be enough. However, capturing more features of the dependence structure might be advantageous in order to improve the approximation quality of the finite sample distributions.

Bootstrap procedures are typically carried out individually according to the dependence structure of the time series. There exists no unique way. Each procedure has its specific application situation, drawbacks and advantages. For an overview of the variety of bootstrap methods, see Politis and Romano (1996), Härdle et al. (2003), Politis (2003), Paparoditis and Politis (2009), Kreiss and Paparoditis (2011), Kreiss and Lahiri (2012), or the references therein.

In the following, we introduce some types of well-established bootstrap for dependent data.

- **Residual bootstrap**: The main idea of residual bootstrap procedures is to fit a parametric model (e.g. AR process) to the data and to apply the classical i.i.d. bootstrap to the estimated residuals, which are assumed to be (at leaset approximately) i.i.d. random variables. Bootstrap procedures of this type usually show a quite good performance in simulations. They are parametric and only applicable for some specific time series models because of the model fitting. One of the most popular methods is the AR sieve bootstrap (see i.e. Kreiss (1988, 1997), Paparoditis and Streitberg (1992) and Bühlmann(1997)), which works for the time series with autoregressive representation. A natural extension of the AR sieve would be an MA sieve, for which MA models need to be fitted, and this model fitting is relative difficult to be done. As an alternative to the idea of MA sieve, McMurry and Politis (2010) proposed the so-called linear process bootstrap based on the estimate of autocovariance of the underlying stationary time series. This procedure works under certain assumptions for the time series with $MA(\infty)$ representation.

- **Block bootstrap**: The Block bootstrap is probably the most straight generalization of the i.i.d. bootstrap for dependent data. The observed sample of size n is divided in m blocks with $m << n$ and $m \to \infty$ as $n \to \infty$. Under the assumption of strict stationarity of the underlying time series, the blocks are i.i.d. The bootstrap pseudo-observations are obtained by drawing randomly with replacement from this set of blocks. The dependence structure of the original data is captured by the neighboring observations within a block. Different versions of block bootstrap have been proposed, i.e. the nonoverlapping block bootstrap, the moving block bootstrap, the stationary bootstrap and the tapered block bootstrap. For references see Carlstein (1986), Künsch (1989), Liu and Singh (1992), Politis and Romano (1992, 1994), Bühlmann and Künsch (1995) and Paparoditis and Politis (2001), among others. The bootstrap methods of this type are nonparametric, and work under very weak conditions on the dependence structure. But they usually do not perform as well as the parametric methods.

- **Dependent wild bootstrap**: The main idea of the wild bootstrap, proposed by Wu (1986), is to generate the bootstrap pseudo-observations by multiplying each centered fitted residual by an i.i.d. external random variable with zero mean and unit variance. The dependent wild bootstrap, introduced by Shao (2010), extends the wild bootstrap of Wu to the time series with dependent setting. The bootstrap pseudo-observations are generated by multiplying each centered original observation of the time series by an external random variable, which comes from a stationary process with zero mean, unit variance and covariance, which is a kernel function. This is a nonparametric approach as well. Comparison to some block bootstrap procedures is given by Shao (2010).

- **Frequency domain bootstrap**: Compared to the bootstrap procedures mentioned above, which work in the time-domain, the frequency domain bootstrap methods rely on the asymptotic features of the periodogram. For the periodogram it is known that its values for different frequencies are asymptotically independent. These methods do not require parametric assumptions and mostly show reasonable behavior in simulations. The drawback of these bootstrap methods is that their applicability is restricted to statistics that can be expressed as functionals of the periodogram. No bootstrap replications in the time domain are produced and thus some dependence properties of the underlying time series can not be reproduced. For references see Franke and Härdle (1992), Dahlhaus and Janas (1996), Paparoditis (2002) and Shao and Wu (2007), among others. To overcome the mentioned drawback, hybrid bootstrap procedures based on the combination of a time domain with a frequency domain bootstrap procedure have been proposed. For references see Kreiss and Paparoditis (2003), Jentsch and Kreiss (2010) and Kreiss and Paparoditis (2012), among others.

- **Local bootstrap**: Local bootstrap procedures generate the bootstrap pseudo-observations from a neighborhood of each data point. They are useful for time series processes with underlying varying trend function, i.e. nonparametric regression model, local stationary process. For references see Shi (1991), Paparoditis and Politis (2000), among others. This idea can be combined with other bootstrap procedures. One of them is the so-called local block bootstrap, introduced by Paparoditis and Politis (2002) and Dowal et al. (2003) (see Dowal et al. (2013) for application for trending time series). The bootstrap pseudo-observations are generated by drawing randomly with replacement from the blocks, which are close to the data point. A local version of the dependent wild bootstrap will be proposed later for our approach.

In this thesis, we discuss i.i.d bootstrap, wild bootstrap and linear process bootstrap for a univariate nonparametric model with independent innovations; linear process bootstrap for a univariate nonparametric model with weak dependence; local bootstrap for a multivariate independent model; local block bootstrap and local dependent wild bootstrap for a multivariate weakly dependent model. All models are nonstationary. We use E^* and Var^* to denote the bootstrap expected value and variance, conditional on a realization of the original time series, as they are used in the literature.

1.4 Main results of this thesis

In this thesis we discuss the application of bootstrap methods in the area of financial market volatility estimation. Discrete-time models for the log-returns with underlying time varying volatility functions are proposed, and classical asymptotic theory for realized volatility estimators based on high frequency intraday data is given. To improve upon the first order asymptotic theory, we propose bootstrap methods individually according to different model assumptions. The asymptotic validity of the proposed procedures is proved, and the finite sample properties of the proposals are investigated in simulation studies.

The remainder of this thesis is organized as follows.
In chapter 2, we deal with a univariate time varying setup, in which the innovations are independent and identically distributed. We propose a kernel estimator to estimate the underlying volatility function and then the noise innovations. Based on the estimated noise innovations we propose a nonparametric i.i.d. bootstrap procedure by resampling these noise innovations, and a nonparametric wild bootstrap procedure by generating pseudo-noise that imitates correctly the first and second order properties of the underlying noise, in order to approximate the distribution of realized bipower variation.
In chapter 3, we have a univariate model with weakly dependent innovation. To mimic the dependence structure of the underlying innovation is challenging. To do this, we make use of the linear process bootstrap method with help of the kernel estimation. The validity of this proposal depends on how accurate the kernel estimator

is. For the kernel estimator, introduced in chapter 1, a condition on the underlying innovation process to guarantee the validity is given.
In chapter 4, we consider multivariate time varying models with independent and with weakly dependent innovations. Asymptotic distributions of a multivariate volatility estimator, named realized covariance are given. In the multivariate setup the varying volatility matrix cannot be estimated any more via nonparametric statistics. That means the nonparametric bootstrap ideal is impracticable without knowing the structure of the volatility matrix. Although we cannot estimate the volatility matrix, we know it varies slowly, and the log-return process is nearly i.i.d in a local time interval. Based on this property, we make use of the local bootstrap for the model with independent innovations, and local block bootstrap for the model with weakly dependent innovations. For the weakly dependent setup, we propose a local version of the wild dependent bootstrap to mimic the varying weak dependence structure as well.

2 Bootstrapping realized volatility with independent innovations

Based on: Feng, G. and Kreiss, J.-P. (2014):
Bootstrapping Realized Bipower Variation.
In: Topics in Nonparametric Statistics, Springer Proceedings in Mathematics and Statistics, Vol. 74, 85-93.

Abstract. Realized bipower variation is often used to measure volatility in financial markets with high frequency intraday data. Considering a nonparametric volatility model in discrete time, we propose a nonparametric i.i.d. bootstrap procedure by resampling the noise innovations based on discrete time returns, and a nonparametric wild bootstrap procedure by generating pseudo-noise that imitates correctly the first and second order properties of the underlying noise, in order to approximate the distribution of realized bipower variation. Asymptotic validity of the proposed procedures is proved. Furthermore, the finite sample properties of the proposals are investigated in a simulation study and are also compared with the standard normal approximation.

2.1 Introduction

We begin with a standard continuous-time model for the log-price process (P_t) of a financial asset, which is given by

$$dP_t = \mu_t dt + \sigma_t dW_t, \tag{2.1}$$

where μ_t denotes the drift, σ_t is a volatility term, and W_t is a standard Brownian motion. Assume that equidistant intraday data with lag $1/n$, $n \in \mathbb{N}$ is observable.

Then

$$X_{i,n} := P_{\frac{i}{n}} - P_{\frac{i-1}{n}}$$

denotes the intraday log-return over the time interval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$.
The integrated volatility (IV) over a day, an important value to quantify the variation of the price process, is defined as

$$IV := \int_0^1 \sigma_t^2 dt.$$

Encouraged by the increased availability of high frequency data, there exists a quite large number of publications that deal with the estimation of integrated volatility in the last few years (see e.g. Andersen and Bollerslev (1998), Barndorff-Nielsen and Shephard (2002), Andersen and Bollerslev (2004), Barndoff-Nielsen et al (2006)). The microstructure noise effect of high frequency data on the properties of estimators of integrated volatility is observed, but will not be considered in this thesis.
A simple estimator of IV, known as realized volatility (RV), is defined for the model above as

$$RV := \sum_{i=1}^{n} X_{i,n}^2.$$

A Central Limit Theorem for $\sqrt{n}(RV - IV)$ is for example given by Barndorff-Nielsen and Shephard (2002).
The more general realized bipower variation (RBV) estimator (see e.g. Barndorff-Nielsen and Shephard (2004a)) is defined as

$$RBV(r,s) := n^{\frac{r+s}{2}-1} \sum_{i=2}^{n} |X_{i,n}|^r |X_{i-1,n}|^s, \quad r,s \geq 0.$$

Barndoff-Nielsen et al. (2006) showed the following convergence in probability

$$RBV(r,s) \xrightarrow{p} \mu_r \mu_s \int_0^1 |\sigma_u|^{r+s} du,$$

and under certain assumptions on the stochastic volatility process (σ_t), as $n \to \infty$,

$$\widetilde{T}_n := \frac{\sqrt{n}\left(RBV(r,s) - \mu_r \mu_s \int_0^1 |\sigma_t|^{r+s} dt\right)}{\rho(r,s)} \xrightarrow{d} N(0,1), \tag{2.2}$$

where $\mu_r = E(|u|^r)$, $u \sim N(0,1)$ and

$$\rho^2(r,s) = \left(\mu_{2r}\mu_{2s} + 2\mu_r\mu_s\mu_{r+s} - 3\mu_r^2\mu_s^2\right) \int_0^1 |\sigma_t|^{2(r+s)} dt.$$

They also show that RBV is robust to finite activity jumps if $r, s < 2$.
It is worthwhile to mention that both of the Central Limit Theorems above are conditioning on the path of σ.

As an alternative tool to the first-order asymptotic theory, Goncalves and Meddahi (2009) primarily introduced two bootstrap methods in the context of realized volatility. Podolskij and Ziggel (2007) extended it to realized bipower variation. Podolskij and Ziggel (2007) proved first-order asymptotic validity and used Edgeworth expansions and Monte Carlo simulations to compare the accuracy of the bootstrap with existing approaches. It is worth mentioning that Podolskij and Ziggel as well as Goncalves and Meddahi focus on standardized quantities like (2.2) for their bootstrap procedures. In this section, we propose two further bootstrap methods in the context of a nonparametric model and we do not restrict to standardized quantities.

2.2 Model and assumptions

We consider a discrete-time model for the intraday log-return process $(X_{t,n})$:

$$X_{t,n} := \frac{1}{\sqrt{n}} \sigma \left(\frac{t}{n} \right) \varepsilon_t, \quad t = 1, \ldots, n, \tag{2.3}$$

where $n \in \mathbb{N}$ is the number of intraday observations.

Assumption.
(A1) *σ denotes a spot volatility term. We assume it can be described with a nonstochastic continuous twice differentiable function $\sigma : [0,1] \to (0, \infty)$, which is bounded away from zero.*
(A2) *(ε_t) are i.i.d. but not necessarily normally distributed random variables with $E\varepsilon_t = 0$, $E\varepsilon_t^2 = 1$, $E\varepsilon_t^4 < \infty$.*

A Central Limit Theorem based on this discrete-time model (compare Barndorff-Nielsen and Shephard (2006)) is given as follows.

Theorem 2.1. *For the discrete-time model* (2.3)*, it holds under assumptions (A1) and (A2) and $r, s \leq 2$, as $n \to \infty$, that*

$$T_n(r,s) := \sqrt{n} \left(RBV(r,s) - \mu_r \mu_s \int_0^1 \sigma^{r+s}(u) du \right) \xrightarrow{d} \mathcal{N} \left(0, \widetilde{\rho}^2(r,s) \right), \tag{2.4}$$

where $\mu_r = E(|\varepsilon_1|^r)$ *and*

$$\widetilde{\rho}^2(r,s) = \left(\mu_{2r}\mu_{2s} + 2\mu_r\mu_s\mu_{r+s} - 3\mu_r^2\mu_s^2\right)\int_0^1 \sigma^{2(r+s)}(t)dt.$$

Remark 2.2. *According to the continuous-time model given by* (2.1)*, we could add a drift term to the discrete-time model above, such as:*

$$X_{t,n} := \frac{1}{\sqrt{n}}\sigma\left(\frac{t}{n}\right)\varepsilon_t + \int_{t-1}^{t}\mu\left(\frac{u}{n}\right)du, \quad t = 1,\dots,n, \tag{2.5}$$

where μ *is a non-stochastic continuous function* $\mu : [0,1] \to (0,\infty)$*. The drift term is of order* $1/n$*, which is smaller than the order* $1/\sqrt{n}$ *of the volatility term, has therefore no influence of the asymptotic results for IV estimators based on the high frequent data. For simplicity, we assume that* $\mu(t/n) = 0$ *for all* t *and consider the discrete-time model given by* (2.3) *in the following sections.*

Remark 2.3. *Podolskij and Ziggel (2007) approximate the finite sample distribution of* $\widetilde{T}_n$ *given by* (2.2)*, which is a standardized statistic. We want to approximate the finite sample distribution of* T_n *including its (asymptotic) variance. Thus, if we want to construct a confidence interval of RBV, our results will directly lead to confidence intervals without further estimation of a standard deviation* ρ *(as which has to be done in Podolskij and Ziggel (2007)). We concentrate in the following bootstrap algorithms on* $RBV(1,1)$*. With suitable conditions on the moments of the innovations, similar bootstrap proposals for* $RBV(r,s)$*,* $r,s < 2$ *could be given.*

2.3 Kernel estimator of the spot volatility

Let realizations $X_{1,n},\cdots,X_{n,n}$ be given. To estimate the varying spot volatility structure, we propose a nonparametric estimator:

$$\widehat{\sigma}^2(u) = \begin{cases} \widehat{\sigma}^2\left(h + \frac{1}{n}\right) & u \in \left[0, h + \frac{1}{n}\right) \\ \frac{1}{h}\sum_{t=1}^{n} X_{t,n}^2 K\left(\frac{\frac{t}{n} - u}{h}\right) & u \in \left[h + \frac{1}{n}, 1-h\right] \\ \widehat{\sigma}^2(1-h) & u \in (1-h, 1] \end{cases} \tag{2.6}$$

where $h > 0$ denotes the bandwidth which fulfills $h \to 0$ and $nh \to \infty$ as $n \to \infty$, and $K(u)$ is a probability density (typically with bounded support) which is called kernel function. We propose to use a symmetric kernel function, which is defined in the interval $[-1, 1]$, e.g. the so-called Epanechnikov kernel or Biweight kernel. The estimator $\widehat{\sigma}^2$ is constant in the left boundary area $\left[0, h + \frac{1}{n}\right)$ with the value of $\widehat{\sigma}^2\left(h + \frac{1}{n}\right)$ as well as in the right boundary area $(1-h, 1]$ with the value $\widehat{\sigma}^2(1-h)$. Therefore, we have the uniform consistency of $\widehat{\sigma}$ not only in $\left[h + \frac{1}{n}, 1-h\right]$ but also in $[0, 1]$. For the uniform consistency of $\widehat{\sigma}$, an additional assumption on the noise innovations is given in *(A3)*:

Assumption.
(A3) *$E\left(\exp(a\varepsilon_1^2)\right) \leq C$ for some constants $a > 0$ and $C < \infty$.*

With this assumption the convergence rate of $\max_{i \in \{1,\ldots,n\}} \varepsilon_i^2$ can be estimated, which is useful for the proof of the uniform consistency.

The uniform consistency of the nonparametric spot volatility estimator is given as follows:

Lemma 2.4. *Let $\{X_{t,n} : t = 1, \ldots, n\}$ be given by the discrete-time model* (2.3) *and let assumptions (A1)-(A3) be fulfilled, then for the kernel estimator given by* (2.6) *it holds:*

$$\sup_{i \in \{0,1,\cdots,n\}} \left| \widehat{\sigma}^2\left(\frac{i}{n}\right) - \sigma^2\left(\frac{i}{n}\right) \right| = o_P(1).$$

The integrated volatility can be estimated using this kernel estimator.

Lemma 2.5. *Let $\{X_{t,n} : t = 1, \ldots, n\}$ be given by the discrete-time model* (2.3) *and let assumptions (A1)-(A3) be fulfilled, then for the kernel estimator given by* (2.6) *it holds for $r, s \leq 2$, as $n \to \infty$, that:*

$$\frac{1}{n-1} \sum_{i=2}^{n} \widehat{\sigma}^r\left(\frac{i}{n}\right) \widehat{\sigma}^s\left(\frac{i-1}{n}\right) \xrightarrow{p} \int_0^1 \sigma^{r+s}(u)du.$$

2.4 Nonparametric i.i.d. bootstrap

The i.i.d. bootstrap for realized volatility introduced by Goncalves and Meddahi (2009) is motivated from constant volatility, i.e. they used a standard resampling scheme from observed log-returns and showed the asymptotic validity under certain assumptions. Podolskij and Ziggel (2007) introduced a bootstrap method with the similar idea in context of bipower variation. In contrast, we propose a nonparametric bootstrap procedure by resampling estimated noise innovations based on discrete time returns, which closely mimics the varying volatility structure of observed log-returns.

Bootstrap Procedure
Let realizations $X_{1,n}, \cdots, X_{n,n}$ be given. The nonparametric i.i.d. bootstrap algorithm is precisely described by the following steps.

- **Step 1:** Compute $\widehat{\sigma}$ via the kernel estimator, given by (2.6).
- **Step 2:** Let $\widehat{\varepsilon}_t = \dfrac{\sqrt{n}X_{t,n}}{\widehat{\sigma}\left(\frac{t}{n}\right)}$, $t = 1, \cdots, n$. Standardizing $\{\widehat{\varepsilon}_1, \cdots, \widehat{\varepsilon}_n\}$ gives $\{\overline{\varepsilon}_1, \cdots, \overline{\varepsilon}_n\}$.
- **Step 3:** Generate the bootstrap intraday returns via

$$X^*_{t,n} = \frac{1}{\sqrt{n}}\widehat{\sigma}\left(\frac{t}{n}\right)\varepsilon^*_t,$$

in which $\varepsilon^*_t = \overline{\varepsilon}_{I_t}$, $I_t \sim$ Laplace on $\{1, \cdots, n\}$, i.e., the ε^*_t is drawn with replacement from the set $\{\overline{\varepsilon}_1, \cdots, \overline{\varepsilon}_n\}$.

The bootstrap realized bipower variation is defined as:

$$RBV^*(r,s) := n^{\frac{r+s}{2}-1}\sum_{i=2}^{n} |X^*_{i,n}|^r |X^*_{i-1,n}|^s, \quad r, s \geq 0.$$

Validity of the Bootstrap
With the estimated noise innovation, the first and second order properties of the underlying noise can be correctly imitated.

Lemma 2.6. *Let $\{X_t : t = 1, \dots, n\}$ be given by the discrete-time model* (2.3) *and the assumptions (A1)-(A3) be fulfilled. Let $\{\varepsilon^*_t : t = 1, \dots, n\}$ be generated via the nonparametric i.i.d. bootstrap algorithm as described above with a kernel estimator $\widehat{\sigma}$, for which $\sup_{i\in\{0,1,\cdots,n\}} |\widehat{\sigma}^2(i/n) - \sigma^2(i/n)| = o_p(1)$. Then the following hold:*

$$\sup_{t\in\{1,\dots,n\}} \{E^*|\varepsilon^*_t| - E|\varepsilon_t|\} = o_p(1)$$
$$\sup_{t\in\{1,\dots,n\}} \left\{E^*|\varepsilon^*_t|^2 - E|\varepsilon_t|^2\right\} = o_p(1).$$

Theorem 2.7. *Under the same assumptions of Lemma 2.6, it holds, as $n \to \infty$, that*

$$T_n^*(1,1) := \sqrt{n}\left(RBV^*(1,1) - \widehat{\mu}_1^2 \frac{1}{n}\sum_{t=2}^{n} \widehat{\sigma}\left(\frac{t}{n}\right)\widehat{\sigma}\left(\frac{t-1}{n}\right)\right) \xrightarrow{d} \mathcal{N}(0, \widetilde{\rho}^2(1,1)) \quad (2.7)$$

in probability. Here $\widehat{\mu}_r = E^*|\varepsilon_1^*|^r = \frac{1}{n}\sum_{i=1}^{n} |\overline{\varepsilon}_i|^r$. $\widetilde{\rho}^2(1,1)$ *is defined in Theorem 2.1. The result implies as $n \to \infty$,*

$$\sup_{x\in\mathbb{R}} |P(T_n^*(1,1) \leq x) - P(T_n(1,1) \leq x)| \xrightarrow{p} 0.$$

2.5 Nonparametric wild bootstrap

Based on the wild Bootstrap for realized volatility introduced by Goncalves and Meddahi (2009), a wild bootstrap method in context of bipower variation was developed by Podolskij and Ziggel (2007). It uses the same summands as the original realized bipower variation, but the returns are all multiplied by an external random variable. We propose a nonparametric wild bootstrap procedure by generating pseudo-noise that imitates correctly the first and second order properties of the underlying noise.

Bootstrap Procedure

Let realizations $X_{1,n}, \cdots, X_{n,n}$ be given. The nonparametric wild bootstrap algorithm is precisely described by the following steps.

- **Step 1:** Compute $\widehat{\sigma}$ with the kernel estimator given by (2.6).

- **Step 2:** Generate pseudo-noise $\varepsilon_1^*, \cdots, \varepsilon_n^*$ with ε_t^* i.i.d. such that $E^*\varepsilon_t^* = 0$,

  $$E^*|\varepsilon_t^*| = \sqrt{\frac{RBV(1,1)}{\frac{1}{n}\sum_{i=1}^{n}\widehat{\sigma}^2\left(\frac{i}{n}\right)}} \quad \text{and} \quad E^*|\varepsilon_t^*|^2 = \sqrt{\frac{RBV(2,2)}{\frac{1}{n}\sum_{i=1}^{n}\widehat{\sigma}^4\left(\frac{i}{n}\right)}}.$$

  For example, one can easily define even a two point distribution that matches all three moment conditions.

- **Step 3:** Generate the wild bootstrap intraday returns via

  $$X_{t,n}^{WB} = \frac{1}{\sqrt{n}}\widehat{\sigma}\left(\frac{t}{n}\right)\varepsilon_t^*.$$

The bootstrap realized bipower variation is defined as:

$$RBV^{WB}(r,s) := n^{\frac{r+s}{2}-1} \sum_{i=2}^{n} |X_{i,n}^{WB}|^r |X_{i-1,n}^{WB}|^s, \quad r,s \geq 0.$$

Validity of the Bootstrap

Based on the result $RBV(r,r) \xrightarrow{p} (E|\varepsilon|^r)^2 \int_0^1 \sigma^{2r}(u)du$ for our model (compare Barndorff-Nielsen and Shephard (2004a) for stochastic volatility), we have that

$$E^*|\varepsilon_t^*|^r \xrightarrow{p} E|\varepsilon_t|^r, \qquad r = 1, 2,$$

so that the first and second order properties of the underlying noise are correctly imitated.

Theorem 2.8. *Let $\{X_t : t = 1, \ldots, n\}$ be given by the discrete-time model* (2.3) *and the assumptions (A1)-(A3) be fulfilled. Let $\{\varepsilon_t^* : t = 1, \ldots, n\}$ be estimated via the nonparametric wild bootstrap algorithm with a kernel estimator $\widehat{\sigma}$, for which $\sup_{i \in \{0,1,\cdots,n\}} |\widehat{\sigma}^2(i/n) - \sigma^2(i/n)| = o_p(1)$. Then it holds that as $n \to \infty$,*

$$\begin{aligned} T_n^{WB}(1,1) := \sqrt{n} \left(RBV^{WB}(1,1) - (E^*|\varepsilon_1^*|)^2 \frac{1}{n} \sum_{t=1}^{n} \widehat{\sigma}\left(\frac{t}{n}\right) \widehat{\sigma}\left(\frac{t-1}{n}\right) \right) \\ \xrightarrow{d} \mathcal{N}(0, \widetilde{\rho}^2(1,1)) \end{aligned} \tag{2.8}$$

in probability. $\widetilde{\rho}^2(1,1)$ is defined in Theorem 2.1. This implies as $n \to \infty$,

$$\sup_{x \in \mathbb{R}} |P(T_n^{WB}(1,1) \leq x) - P(T_n(1,1) \leq x)| \xrightarrow{p} 0.$$

2.6 A simulation study

We compare the accuracy of the proposed bootstrap methods with the normal approximation by considering 2.5% and 97.5% quantiles of the finite sample statistic T_n by Using Monte Carlo simulations.

We choose here the volatility function $\sigma(u) = 0.32(u - 0.5)^2 + 0.04$ (cf. panel 1, Figure 2.1), a noise innovation of $\varepsilon \sim N(0,1)$ and a sample size of $n = 200$. The observations are simulated according to model (2.3) and T_n is computed. On one hand, we compute $\widehat{\sigma}$ with the kernel estimator (2.6), generate the bootstrap data and compute T_n^* and T_n^{WB} (cf. (2.7) and (2.8)). With 1000 repetitions of the bootstrap procedures, we get the empirical quantiles of the distribution of T_n^* and T_n^{WB}. On

the other hand, we computed the desired quantiles via normal approximation with estimated standard deviation $\widetilde{\rho}(1,1)$.
The whole simulation is repeated 1000 times to obtain boxplots of sample quantiles of interest. The boxplots on the left side of panel 2 in Figure 2.1 give the approximations via nonparametric i.i.d. bootstrap, while the ones in the middle give the approximations via nonparametric wild bootstrap. The boxplots on the right side present the results obtained from normal approximation. The true quantiles, indicated as lines in panel 2 of Figure 2.1, of the finite sample distribution of T_n are also obtained by simulation (100.000 repetitions).

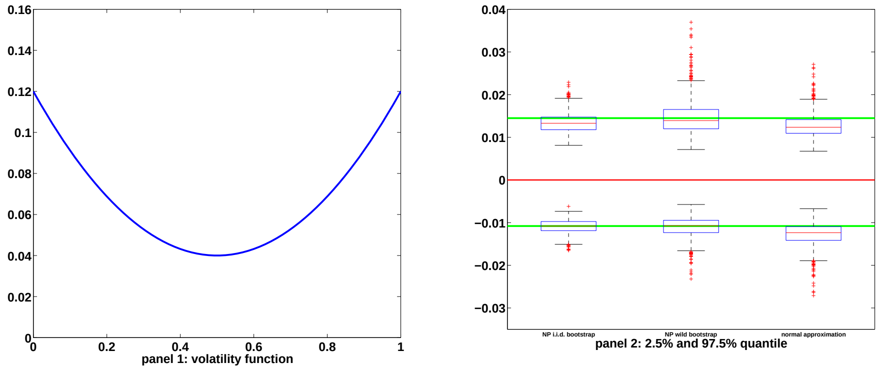

Figure 2.1: Volatility function and quantiles of nonparametric bootstrap and normal approximation

One can see that the medians of both bootstrap boxplots nearly hit the true 2.5% quantile. For the 97.5% quantile, both bootstraps perform not as well as for the 2.5% quantile, but at least slightly better than the normal approximation. One reason could be the fact that the kernel volatility estimator (2.6) underestimates the true high volatility at the boundary of the interval. Therefore, the 97.5% quantile, which is strongly related to high volatility, is not so well approximated. The normal approximation of course can not mimic the skewness of the finite sample distribution of T_n.

With the same setup of n and ε, but another volatility function, namely $\sigma(u) = 0.08 + 0.04 \sin(2\pi u)$ (cf. panel 1, Figure 2.2), we do the same simulation study as before. The results are displayed in panel 2 of Figure 2.2.
The situation is quite similar. Bootstrap medians are closer to the true values than the medians of the normal approximation, indicating that bootstrap might be better able to mimic to a certain extent the skewness of the distribution of T_n. An under-

estimation of the 97.5% quantile does not appear in this case, in which the high volatility is located in a non-border area of the interval. Both bootstraps therefore perform better for the 97.5% quantile in contrast with the simulation before.

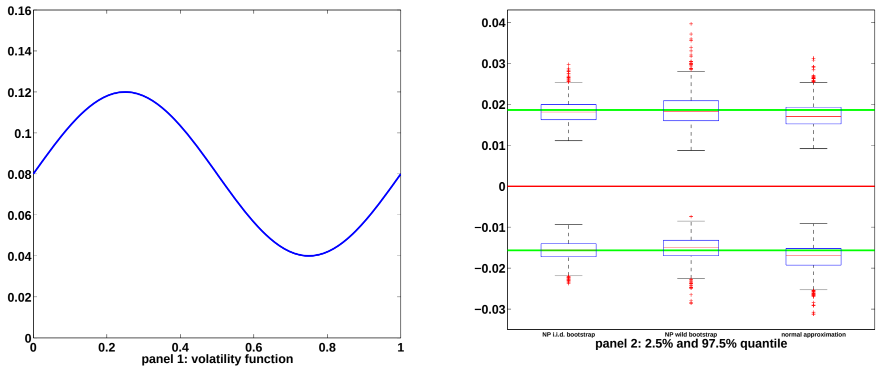

Figure 2.2: Volatility function and quantiles of nonparametric bootstrap and normal approximation

2.7 Proofs and auxiliary results

To simplify the notation, we use σ_t instead of $\sigma\left(\frac{t}{n}\right)$ and $\widehat{\sigma}_t$ instead of $\widehat{\sigma}\left(\frac{t}{n}\right)$ in the following proofs.

Proof of Theorem 2.1

We abbreviate

$$U_{i,n} := \sigma_i^r \sigma_{i-1}^s \left(|\varepsilon_i|^r |\varepsilon_{i-1}|^s - E|\varepsilon_1|^r E|\varepsilon_1|^s\right)$$

and observe that for each $n \in \mathbb{N}$, $\{U_{i,n} : i = 1, \cdots, n\}$ are centered and 1-dependent random variables. $T_n(r,s)$ can be written as follows: $T_n(r,s) = A_n + B_n$, where

$$A_n := \frac{1}{\sqrt{n}} \sum_{i=2}^{n} U_{i,n}$$

and

$$B_n := \frac{1}{\sqrt{n}} \sum_{i=2}^{n} \sigma_i^r \sigma_{i-1}^s E|\varepsilon_1|^r E|\varepsilon_1|^s - \sqrt{n} E|\varepsilon_1|^r E|\varepsilon_1|^s IV.$$

To handle A_n, we make use of a Central Limit Theorem for on average stationary m-dependent triangular arrays (cf. Kreiss(1997)). It can be shown, as $n \to \infty$, that

$$\frac{1}{n-1} \sum_{i=2}^{n} E\left(U_{i,n}^2\right) \quad = \quad \frac{1}{n-1} \sum_{i=2}^{n} \sigma_i^{2r} \sigma_{i-1}^{2s} \left(E|\varepsilon_i|^r E|\varepsilon_{i-1}|^s - \left(E|\varepsilon_1|^r E|\varepsilon_1|^s\right)^2\right)$$

$$
\begin{aligned}
&= \frac{1}{n}\sum_{i=2}^{n}\sigma_i^{2r}\left(\sigma_i+\mathcal{O}\left(\frac{1}{n}\right)\right)^{2s}\left(E|\varepsilon_i|^r E|\varepsilon_{i-1}|^s-(E|\varepsilon_1|^r E|\varepsilon_1|^s)^2\right)\\
&\longrightarrow \left(E|\varepsilon_i|^r E|\varepsilon_{i-1}|^s-(E|\varepsilon_1|^r E|\varepsilon_1|^s)^2\right)\int_0^1\sigma^{r+s}(u)du := c(0),
\end{aligned}
$$

and in the same way

$$
\frac{1}{n}\sum_{i=2}^{n-1}E\left(U_{i,n}U_{i+1,n}\right)\longrightarrow\left(E|\varepsilon_1|^r E|\varepsilon_1|^{s+r}E|\varepsilon_1|^s-(E|\varepsilon_1|^r E|\varepsilon_1|^s)^2\right)\int_0^1\sigma^{r+s}(u)du := c(1).
$$

For $h>1, h\in\mathbb{N}$, the independence of $U_{i,n}$ and $U_{i+h,n}$ leads to

$$
\frac{1}{n}\sum_{i=2}^{n-1}E\left(U_{i,n}U_{i+h,n}\right)=0 := c(h).
$$

The function $c(\cdot)$ fullfills

$$
c(0)+2\sum_{h=1}^{\infty}c(h)=\tilde{\rho}^2(r,s).
$$

and we have, as $n\to\infty$, that

$$
Var(T_n(r,s))\longrightarrow\tilde{\rho}^2(r,s).
$$

Recalling assumption *(A2)*, that $(\varepsilon_t)_{t=1,\dots,n}$ are i.i.d. and $E\varepsilon_1^4<\infty$, and assumption *(A1)*, that σ is bounded (say by a constant C), and let $\zeta>0$, a computation furthermore leads to

$$
\begin{aligned}
&\lim_{n\to\infty}\sum_{i=2}^{n}E\left(\frac{1}{n-1}U_{i,n}^2\mathbb{1}_{\{|U_{i,n}|>\zeta\sqrt{n}\}}\right)\\
\leq\;&\lim_{n\to\infty}\frac{1}{n-1}\sum_{i=2}^{n}E\Big(C^{2r+2s}\left(|\varepsilon_i|^r|\varepsilon_{i-1}|^s-E|\varepsilon_1|^r E|\varepsilon_1|^s\right)^2\\
&\mathbb{1}_{\{|C^{r+s}(|\varepsilon_i|^r|\varepsilon_{i-1}|^s-E|\varepsilon_1|^r E|\varepsilon_1|^s)|>\zeta\sqrt{n}\}}\Big)\\
=\;&\lim_{n\to\infty}C^{2r+2s}E\left(\left(|\varepsilon_2|^r|\varepsilon_1|^s-E|\varepsilon_1|^r E|\varepsilon_1|^s\right)^2\mathbb{1}_{\{|C^{r+s}(|\varepsilon_2|^r|\varepsilon_1|^s-E|\varepsilon_1|^r E|\varepsilon_1|^s)|>\zeta\sqrt{n}\}}\right)=0.
\end{aligned}
$$

Thus a Lindeberg condition is also fullfilled and Lemma 8.4, Kreiss (1997), gives

$$
A_n\xrightarrow{d}\mathcal{N}\left(0,\tilde{\rho}^2(s,r)\right).
$$

Additionally, it is easy to show that, as $n\to\infty$, $B_n\to 0$, which concludes the proof of Theorem 2.1 due to Slutsky's Lemma. □

Proofs for the uniform consistency of the kernel estimator

Lemma 2.9 below is required for proof of Lemma 2.4.

Lemma 2.9. *Let assumption (A2) and (A3) be fulfilled, it holds*

$$\max_{1\le i\le n} \varepsilon_i^2 = \mathcal{O}_p\left(\log(n)\right)$$

Proof. It can be shown under the assumption $E\left(\exp(a\varepsilon_1^2)\right) \le C$, that:

$$\begin{aligned} E\left(\max_{1\le i\le n} \varepsilon_i^2\right) &= \frac{1}{a} E\left(\log\left(\max_{1\le i\le n} \exp\left(a\varepsilon_i^2\right)\right)\right) \\ &\le \frac{1}{a}\log\left(E\left(\max_{1\le i\le n} \exp\left(a\varepsilon_i^2\right)\right)\right) \\ &\le \frac{1}{a}\log\left(E\left(\sum_{i=1}^{n} \exp\left(a\varepsilon_i^2\right)\right)\right) \\ &\le \frac{1}{a}\log(Cn), \end{aligned}$$

which implies the desired result.

Proof of Lemma 2.4

We prove at first the uniform consistency for $\left[h+\frac{1}{n}, 1-h\right]$ and then consider the situations for the Boundary areas. Note that the kernel function $K(u)$ we proposed is symmetric. With a Taylor series expansion, it is easy to show the pointwise consistency for $u \in \left[h+\frac{1}{n}, 1-h\right]$, such as

$$\widehat{\sigma}^2(u) = \sigma^2(u) + \mathcal{O}_p(h^2) + \mathcal{O}_p\left(\frac{1}{nh}\right).$$

To prove the uniform consistency, we construct subgroups of the sample data $\{X_{i,n} : \lfloor nh\rfloor + 1 \le i \le n - \lfloor nh\rfloor\}$. Let l denotes the number of the data points in each subgroup with $l = l(n) \to \infty$ as $n \to 0$. For simplicity of notation we want $m = \frac{n}{l}$, which denotes the number of the subgroups, to be positive integer. Let $I_k := \{(k-1)l + 1 + \lfloor hn\rfloor, \ldots, kl + \lfloor hn\rfloor\}, k = 1, \ldots, m$. According to Lemma 2.9, we obtain:

$$\begin{aligned} \max_{1\le k\le m} \sup_{t\in I_k} \left|\widehat{\sigma}_t^2 - \widehat{\sigma}_{kl}^2\right| &= \max_{1\le k\le m} \sup_{t\in I_k} \left|\frac{1}{nh}\sum_{s=1}^{n} \sigma_s^2\varepsilon_s^2 \left(K\left(\frac{s-t}{nh}\right) - K\left(\frac{s-kl}{nh}\right)\right)\right| \\ &\le \max_{1\le k\le m} \sup_{t\in I_k} \left|\frac{1}{nh}(2nh+l) \max_{1\le s\le n} \varepsilon_s^2 \mathcal{O}\left(\frac{l}{nh}\right)\right| \\ &= \frac{1}{nh}(2nh+l)\mathcal{O}_p\left(\log(n)\right)\mathcal{O}\left(\frac{l}{nh}\right) \\ &= \mathcal{O}_p\left(\frac{l\log(n)}{nh}\right). \end{aligned} \tag{2.9}$$

Further we get:

$$
\begin{aligned}
& \sup_{i\in\{1+\lfloor hn\rfloor,\dots,n-\lfloor hn\rfloor\}} \left|\widehat{\sigma}_t^2-\sigma_t^2\right| \\
\leq\ & \max_{1\leq k\leq m}\sup_{t\in I_k}\left|\widehat{\sigma}_t^2-\sigma_t^2-\left(\widehat{\sigma}_{kl}^2-\sigma_{kl}^2\right)\right| + \max_{1\leq k\leq m}\left|\widehat{\sigma}_{kl}^2-\sigma_{kl}^2\right| \\
\leq\ & \max_{1\leq k\leq m}\sup_{t\in I_k}\left|\widehat{\sigma}_t^2-\widehat{\sigma}_{kl}^2\right| + \max_{1\leq k\leq m}\sup_{t\in I_k}\left|\sigma_t^2-\sigma_{kl}^2\right| + \sum_{k=1}^{m}\left|\widehat{\sigma}_{kl}^2-\sigma_{kl}^2\right| \\
=\ & \mathcal{O}_p\left(\frac{l\log(n)}{nh}\right)+\mathcal{O}\left(\frac{l}{n}\right)+\mathcal{O}_p\left(mh^2\right)+\mathcal{O}_p\left(\frac{m}{nh}\right),
\end{aligned}
$$

where $m=\dfrac{n}{l}$. To ensure that as $n\to\infty$, $\dfrac{l\log(n)}{nh}\to 0$, $\dfrac{l}{n}\to 0$, $\dfrac{nh^2}{l}\to 0$ and $\dfrac{1}{lh}\to 0$, l and h can be selected, i.e. $l=n^{\frac{2}{3}}$ and $h=n^{-\frac{1}{5}}$. We have in this case

$$
\sup_{i\in\{1+\lfloor hn\rfloor,\dots,n-\lfloor hn\rfloor\}} \left|\widehat{\sigma}_t^2-\sigma_t^2\right| = \mathcal{O}_p(n^{-\frac{1}{15}}) \tag{2.10}
$$

This yields the uniform consistency in $u\in\left[\dfrac{1}{n}+h,1-h\right]$ for some suitable l,h. For $u\in\left[0,h+\dfrac{1}{n}\right)$, we have

$$
\begin{aligned}
& \sup_{u\in\left[0,h+\frac{1}{n}\right)}\left|\widehat{\sigma}^2(u)-\sigma^2(u)\right| \\
=\ & \sup_{u\in\left[0,h+\frac{1}{n}\right)}\left|\widehat{\sigma}^2\left(h+\frac{1}{n}\right)-\sigma^2\left(h+\frac{1}{n}\right)+\sigma^2\left(h+\frac{1}{n}\right)-\sigma^2(u)\right| \\
\leq\ & \left|\widehat{\sigma}^2\left(h+\frac{1}{n}\right)-\sigma^2\left(h+\frac{1}{n}\right)\right| + \sup_{u\in\left(0,h+\frac{1}{n}\right)}\left|\sigma^2\left(h+\frac{1}{n}\right)-\sigma^2(u)\right| \\
=\ & \mathcal{O}_p\left(h^2\right)+\mathcal{O}_p\left(\frac{1}{nh}\right)+\mathcal{O}\left(h+\frac{1}{n}\right).
\end{aligned}
$$

and analogously, for $u\in(1-h,1]$

$$
\sup_{u\in(1-h,1]}\left|\widehat{\sigma}^2(u)-\sigma^2(u)\right| = \mathcal{O}_p\left(h^2\right)+\mathcal{O}_p\left(\frac{1}{nh}\right)+\mathcal{O}(h),
$$

Thus the uniform consistency in the whole area $[0,1]$ holds for suitable l and h. Furthermore, the computation above gives $\sup_{i\in\{2,\dots,n\}}\left|\widehat{\sigma}_i^2-\widehat{\sigma}_{i-1}^2\right|=\mathcal{O}_p\left(\dfrac{\log(n)}{nh}\right)$. Due to the mean value theorem, we have

$$
\sup_{t\in\{2,\dots,n\}}\left|\widehat{\sigma}_t-\widehat{\sigma}_{t-1}\right| \leq \sup_{t\in\{2,\dots,n\}}\left|\widehat{\sigma}_t^2-\widehat{\sigma}_{t-1}^2\right|\frac{1}{2\sqrt{\inf_{i\in\{1,\dots,n\}}\widehat{\sigma}_i^2}}
$$

$$\begin{aligned}
&\leq \mathcal{O}_p\left(\frac{\log(n)}{nh}\right)\frac{1}{2\sqrt{\inf_{i\in\{1,\ldots,n\}}\sigma_i^2-\sup_{i\in\{1,\ldots,n\}}|\widehat{\sigma}_i^2-\sigma_i^2|}}\\
&= \mathcal{O}_p\left(\frac{\log(n)}{nh}\right)\frac{1}{2\sqrt{\inf_{i\in\{1,\ldots,n\}}\sigma_i^2-\mathcal{O}_p\left(\frac{\log(n)}{nh}\right)}}\\
&= \mathcal{O}_p\left(\frac{\log(n)}{nh}\right). \qquad (2.11)
\end{aligned}$$

Due to the mean value theorem, we have two further uniform consistencies as follows:

$$\begin{aligned}
\sup_{t\in\{1,\ldots,n\}}|\widehat{\sigma}_t-\sigma_t| &\leq \sup_{t\in\{1,\ldots,n\}}\left|\widehat{\sigma}_t^2-\sigma_t^2\right|\frac{1}{2\sqrt{\inf_{i\in\{1,\ldots,n\}}\widehat{\sigma}_i^2}}\\
&= \sup_{t\in\{1,\ldots,n\}}\left|\widehat{\sigma}_t^2-\sigma_t^2\right|\frac{1}{2\sqrt{\inf_{i\in\{1,\ldots,n\}}\sigma_i^2-\sup_{i\in\{1,\ldots,n\}}|\widehat{\sigma}_i^2-\sigma_i^2|}}\\
&= o_p(1)\frac{1}{2\sqrt{\inf_{i\in\{1,\ldots,n\}}\sigma_i^2-o_p(1)}}\\
&= o_p(1). \qquad (2.12)
\end{aligned}$$

$$\begin{aligned}
\sup_{t\in\{1,\ldots,n\}}\left|\widehat{\sigma}_t^4-\sigma_t^4\right| &\leq \sup_{t\in\{1,\ldots,n\}}\left|\widehat{\sigma}_t^2-\sigma_t^2\right|2\sqrt{\sup_{i\in\{1,\ldots,n\}}\widehat{\sigma}_i^2}\\
&= \sup_{t\in\{1,\ldots,n\}}\left|\widehat{\sigma}_t^2-\sigma_t^2\right|2\sqrt{\sup_{i\in\{1,\ldots,n\}}\sigma_i^2+\sup_{i\in\{1,\ldots,n\}}|\widehat{\sigma}_i^2-\sigma_i^2|}\\
&= o_p(1)2\left(\sup_{i\in\{1,\ldots,n\}}\sigma_i^2+o_p(1)\right)\\
&= o_p(1).
\end{aligned}$$

$\square$

Proof of Lemma 2.5

The uniform consistence given by Lemma 2.4 and (2.12) lead to, for $a,b\leq 2$:

$$\begin{aligned}
\frac{1}{n}\sum_{i=2}^n\widehat{\sigma}_i^a\widehat{\sigma}_{i-1}^b &= \frac{1}{n}\sum_{i=2}^n\left((\widehat{\sigma}_i^a-\sigma_i^a)+\sigma_i^a\right)\left(\left(\widehat{\sigma}_{i-1}^b-\sigma_{i-1}^b\right)+\sigma_{i-1}^b\right)\\
&= \frac{1}{n}\sum_{i=2}^n\sigma_i^a\sigma_{i-1}^b+o_p(1)\\
&= \frac{1}{n}\sum_{i=2}^n\sigma_i^a\left(\left(\sigma_{i-1}^b-\sigma_i^b\right)+\sigma_i^b\right)+o_p(1)\\
&= \frac{1}{n}\sum_{i=2}^n\sigma_i^{a+b}+o(1)+o_p(1)\\
&= \int_0^1\sigma^{a+b}(u)du+o_p(1).
\end{aligned}$$

Together with the finite 4th moment of $\widehat{\sigma}$, we have the desired result. Analogously, it can be easily shown, that for $a, b, c \leq 2$

$$\frac{1}{n-2}\sum_{i=2}^{n-1}\widehat{\sigma}_{i+1}^{a}\widehat{\sigma}_{i}^{b}\widehat{\sigma}_{i-1}^{c} - \int_0^1 \sigma^{a+b+c}(u)du \xrightarrow{p} 0 \tag{2.13}$$

□

Proofs for the nonparametric i.i.d. bootstrap

We show first that the first and second order properties of the underlying noise can be correctly imitated by using the estimated noise innovations.

Proof of Lemma 2.6

The uniform consistency of $\widehat{\sigma}$ (2.12) leads to

$$\sup_{t\in\{0,1,\dots,n\}}\left|\frac{\sigma_t}{\widehat{\sigma}_t}\right| = 1 + o_p(1). \tag{2.14}$$

Let $\widehat{\varepsilon}$ denote the estimated noise innovations. We have $\widehat{\varepsilon}_t = \dfrac{\sigma_t\varepsilon_t}{\widehat{\sigma}_t}$, since $X_t = \dfrac{1}{\sqrt{n}}\sigma_t\varepsilon_t$. It holds then uniformly in t

$$\frac{1}{n}\sum_{t=1}^{n}|\widehat{\varepsilon}_t| = \frac{1}{n}\sum_{t=1}^{n}\left|\frac{\sigma_t}{\widehat{\sigma}_t}\varepsilon_t\right| = \frac{1}{n}\sum_{t=1}^{n}|\varepsilon_t| + o_p(1) \xrightarrow{p} E|\varepsilon_t|. \tag{2.15}$$

According to the nonparametric i.i.d. bootstrap procedure, we have

$$E^*|\varepsilon_t^*| = \frac{1}{n}\sum_{t=1}^{n}|\overline{\varepsilon}_t| = \frac{1}{n}\sum_{t=1}^{n}\left|\frac{\widehat{\varepsilon}_t - \mu_{\widehat{\varepsilon}}}{\sqrt{V_{\widehat{\varepsilon}}}}\right|$$

where $\mu_{\widehat{\varepsilon}}$ is the sample mean and $V_{\widehat{\varepsilon}}$ is the sample covariance of $\{\widehat{\varepsilon}_1, \dots, \widehat{\varepsilon}_n\}$. Using (2.14), it is easy to show that $\mu_{\widehat{\varepsilon}} = o_p(1)$ and $V_{\widehat{\varepsilon}} = 1 + o_p(1)$. Since $E\varepsilon_t^2 < \infty$,

$$Var\left(E^*|\varepsilon_t^*|\right) = \frac{1}{n^2}\sum_{t=1}^{n}Var\left|\frac{\widehat{\varepsilon}_t - \mu_{\widehat{\varepsilon}}}{\sqrt{V_{\widehat{\varepsilon}}}}\right| \longrightarrow 0,$$

and together with (2.15), we have the uniformly consistency of the first order absolute moment:

$$E^*|\varepsilon_t^*| \xrightarrow{p} E|\varepsilon_t|$$

To show the consistency of the second order moment, we use the uniform consistency of $\widehat{\sigma}_t^2$, and follow the same way. $E\varepsilon_t^4 < \infty$ is required respectively. □

Proof of Theorem 2.7

Similarly to the proof of Theorem 2.1, we define for each $n \in \mathbb{N}$ centered and 1-dependent random variables $\{U^*_{i,n} : i = 1, \cdots, n\}$ as

$$U^*_{i,n} := \widehat{\sigma}_i \widehat{\sigma}_{i-1} \left(|\varepsilon^*_i \varepsilon^*_{i-1}| - \widehat{\mu}_1^2 \right),$$

where $\widehat{\mu}_1 = E^*|\varepsilon^*_1| = \frac{1}{n}\sum_{i=1}^{n} |\overline{\varepsilon}_i|$, and rewrite T^*_n as:

$$T^*_n := \frac{1}{\sqrt{n}} \sum_{i=2}^{n} U^*_{i,n},$$

Lemma 2.5, Lemma 2.6 and $E\varepsilon^2 < \infty$ lead to the following results:

$$\begin{aligned} \frac{1}{n}\sum_{i=2}^{n} E^*\left({U^*_{i,n}}^2\right) &= \frac{1}{n}\sum_{i=2}^{n} \widehat{\sigma}_i^2 \widehat{\sigma}_{i-1}^2 \left(\left(E^*|\varepsilon^*_1|^2\right)^2 - \left(E^*|\varepsilon^*_1|\right)^4 \right) \\ &\xrightarrow{p} \left(\left(E|\varepsilon_1|^2\right)^2 - \left(E|\varepsilon_1|\right)^4 \right) \int_0^1 \sigma^4(u)du := c(0). \end{aligned}$$

$$\begin{aligned} \frac{1}{n}\sum_{i=2}^{n-1} E^*\left(U^*_{i,n} U^*_{i+1,n}\right) &= \frac{1}{n}\sum_{i=2}^{n-1} \left| \widehat{\sigma}_{i-1} \widehat{\sigma}_i^2 \widehat{\sigma}_{i+1} \right| \left(\left(E^*|\varepsilon^*_1|\right)^2 E^*|\varepsilon^*_1|^2 - \left(E^*|\varepsilon^*_1|\right)^4 \right) \\ &\xrightarrow{p} \left(\left(E|\varepsilon_1|\right)^2 E|\varepsilon_1|^2 - \left(E|\varepsilon_1|\right)^4 \right) \int_0^1 \sigma^4(u)du := c(1). \end{aligned}$$

For $h > 1, h \in \mathbb{N}$, the independence of $U^*_{i,n}$ and $U^*_{i+h,n}$ leads to

$$\frac{1}{n}\sum_{i=2}^{n-1} E^*\left(U^*_{i,n} U^*_{i+h,n}\right) \xrightarrow{p} 0 := c(h).$$

The function $c(\cdot)$ fullfills

$$c(0) + 2\sum_{h=1}^{\infty} c(h) \xrightarrow{p} \widetilde{\rho}^2(1,1).$$

and we have, as $n \to \infty$, that

$$Var^*(T_n) \xrightarrow{p} \widetilde{\rho}^2(1,1).$$

To prove a Lindeberg condition, we need:

$$\sup_{t\in\{1,\ldots,n\}} \widehat{\sigma}_t \le \sup_{t\in\{1,\ldots,n\}} \sigma_t + \sup_{t\in\{1,\ldots,n\}} |\widehat{\sigma}_t - \sigma_t| = C + o_p(1),$$

where we abbreviate $C = \sup_{t\in\{1,\ldots,n\}} \sigma_t$.

Using the fact that $E^*|\varepsilon^*_1|$ and $E^*|\varepsilon^*_1|^2$ are bounded in probability because of Lemma

2.6 and $\widehat{\sigma}_t$ is uniformly bounded in $t \in \{1,\ldots,n\}$ in probability because of Lemma 2.4, a computation furthermore leads to

$$
\begin{aligned}
& \frac{1}{n-1}\sum_{i=2}^{n} E^*\left(U_{i,n}^{*}{}^2 \mathbb{1}_{\{|U_{i,n}^*|>\zeta\sqrt{n}\}}\right) \\
\leq \quad & \frac{1}{n-1}\sum_{i=2}^{n} E^*\Bigg(\sup_{t\in\{2,\ldots,n\}} (\widehat{\sigma}_t\widehat{\sigma}_{t-1})^2\left(\left|\varepsilon_i^*\varepsilon_{i-1}^*\right| - (E^*|\varepsilon_1^*|)^2\right)^2 \\
& \cdot \mathbb{1}_{\{\sup_{t\in\{2,\ldots,n\}}\left(\widehat{\sigma}_t\widehat{\sigma}_{t-1}\right)\left|\left|\varepsilon_i^*\varepsilon_{i-1}^*\right|-\left(E^*|\varepsilon_1^*|\right)^2\right|>\zeta\sqrt{n}\}}\Bigg) \\
\leq \quad & \sup_{t\in\{1,\ldots,n\}} \widehat{\sigma}_t^4 E^*\left(\left(|\varepsilon_2^*\varepsilon_1^*| - (E^*|\varepsilon_1^*|)^2\right)^2 \mathbb{1}_{\{\sup_{t\in\{1,\ldots,n\}} \widehat{\sigma}_t^2\left|\left|\varepsilon_2^*\varepsilon_1^*\right|-\left(E^*|\varepsilon_1^*|\right)^2\right|>\zeta\sqrt{n}\}}\right) \\
\overset{n\to\infty}{\longrightarrow} \quad & 0 \text{ in probability.}
\end{aligned}
$$

Thus a Lindeberg condition is also fullfilled. The CLT for m-dependent triangular arrays (Lemma 8.4, Kreiss (1997)) yields the desired result. □

Proof of Theorem 2.8

To prove the validity of the nonparametric wild bootstrap procedure, we need to show that the first and second order properties of the underlying noise are correctly imitated.

According to Theorem 2.1, we have

$$RBV(r,r) \xrightarrow{p} (E|\varepsilon_1|^r)^{2r}\int_0^1 \sigma^{2r}(u)du$$

Recalling Step 2 of the nonparametric wild bootstrap procedure and Lemma 2.5, we have for $r = 1, 2$

$$E^*|\varepsilon_t^*|^r = \sqrt{\frac{RBV(r,r)}{\frac{1}{n}\sum_{i=1}^{n}\widehat{\sigma}_i^{2r}}} = \sqrt{\frac{RBV(r,r)}{\int_0^1 \sigma^{2r}(u)du + o_p(1)}} \xrightarrow{p} E|\varepsilon_1|^r.$$

The further proof is quite similar to the one of Theorem 2.7 and is therefore omitted.

3 Bootstrapping realized volatility with weakly dependent innovations

Using a dependent setup for the return process could be more realistic and more challenging. In this section, we consider a discrete-time model with weakly dependent innovations. A standard result of normal approximation for the realized volatility was given. To mimic this asymptotic normal distribution, we need to construct the dependent structure correctly by the bootstrap algorithms. One idea is to use the nonparametric estimator given in the section 2.3 to estimate the varying structure of the spot volatility, and then the innovations. Based on the estimated innovations, the autocovariance of the innovation process can be estimated. Following the linear process bootstrap from McMurry and Politis (2010), a bootstrap normal approximation for the realized volatility can be given. The other ideas, which do not involve nonparametric estimation, such as local block bootstrap and local dependent wild bootstrap, will be discussed in the later section for the multivariate setup.

3.1 Model and assumptions

We consider a discrete-time model for the intraday log-return process $(X_{t,n})$:

$$X_{t,n} := \frac{1}{\sqrt{n}}\sigma\left(\frac{t}{n}\right)\xi_t, \quad t = 1, \dots, n, \tag{3.1}$$

where $n \in \mathbb{N}$ is the number of intraday observations. We make the following assumptions.

Assumption.
(B1) *σ denotes a spot volatility term. We assume, it can be described with a non-stochastic continuous twice differentiable function $\sigma : [0,1] \to (0,\infty)$, which is*

bounded away from zero.
(B2) $(\xi_t)_{t\in\mathbb{Z}}$ *denotes a real-valued stationary time series with* $E\xi_1 = 0$, $E\xi^2 = 1$, $E\xi_1^8 < \infty$ *and has the GMC(2) property.*
(B3) $\left(\xi_t^2\right)_{t\in\mathbb{Z}}$ *is stationary with* $\sum_{h=-\infty}^{\infty} |\gamma_{\xi^2}(h)| < \infty$.
(B4) *For the uniform consistency of the spot volatility estimator, we assume additionally* $E\left(\exp(a\xi_1^2)\right) \leq C$ *for some constants* $a > 0$ *and* $C < \infty$.

Based on the discrete-time model and the assumptions above, the asymptotic distribution of realized volatility is given as follows:

Theorem 3.1. *For the discrete-time model* (3.1)*, it holds under assumptions (B1)-(B3), as* $n \to \infty$*, that*

$$\widetilde{T}_n := \sqrt{n}\left(RV - \int_0^1 \sigma^2(u)du\right) \xrightarrow{d} \mathcal{N}\left(0, \widetilde{V}^2\right), \tag{3.2}$$

where

$$\widetilde{V}^2 := \int_0^1 \sigma^4(u)du \sum_{h=-\infty}^{\infty} \gamma_{\xi^2}(h).$$

3.2 The linear process bootstrap

As an alternative to an MA sieve, McMurry and Politis (2010) proposed the so-called linear process bootstrap. This method is based on the autocovariance estimation, which makes it possible to generate pseudo-observations imitating a MA process without model fitting. Let $\widetilde{\Sigma}_{\kappa,l}^{\epsilon}$ denote the positive definite version of the tapered covariance matrix estimator, which was given by McMurry and Politis (2010). $\kappa(\cdot)$ is the tapered weight function, and l is the banding parameter. If $\kappa(\cdot)$ is chosen to be zero after a point, this bootstrap procedure is suitable for an finite order MA process. By using an $\kappa(\cdot)$ that just trends to zero, but does not equal to zero after a point, the linear process bootstrap is able to imitate a $MA(\infty)$ process. For examples of a weight function and a empirical Rule of picking l see McMurry and Politis (2010) and the references therein.

In the following, we proposes a bootstrap procedure, which is able to capture the weak dependence structure and the underlying varying volatility, by combining the linear process bootstrap with the nonparametric estimation of the varying volatility structure, which is introduced in section 2.3.

Bootstrap procedure

Given realizations $X_{1,n}, \cdots, X_{n,n}$. The bootstrap procedure is described by the following steps.

- **Step 1:** Compute $\widehat{\sigma}$ via the kernel estimator, given by (2.6).
- **Step 2:** Let $\widehat{\xi}_t = \dfrac{\sqrt{n} X_{t,n}}{\widehat{\sigma}\left(\frac{t}{n}\right)}$, $t = 1, \cdots, n$ and $Y_t = \widehat{\xi}_t^2 - \overline{\widehat{\xi}^2}$ for $t = 1, \ldots, n$, where $\overline{\widehat{\xi}^2} = \dfrac{1}{n} \sum_{i=1}^{n} \widehat{\xi}_i^2$. Denote $\underline{Y} = (Y_1, \ldots, Y_n)'$.
- **Step 3:** Let $\underline{W} = \left(\widetilde{\Sigma}_{\kappa,l}^{\epsilon}\right)^{-1/2} \underline{Y}$, $\left(\widetilde{\Sigma}_{\kappa,l}^{\epsilon}\right)^{1/2}$ be the lower triangular matrix associated with the Cholesky decomposition of $\widetilde{\Sigma}_{\kappa,l}^{\epsilon}$, and $\widetilde{\Sigma}_{\kappa,l}^{\epsilon}$ is the positive definite covariance matrix estimator of $\underline{Y}$. Standardizing $\underline{W}$ gives $\underline{Z}$ with elements $\{Z_1, \ldots, Z_n\}$.
- **Step 4:** Generate $Z_1^*, \ldots, Z_n^*$ via the i.i.d. bootstrap based on the sample $Z_1, \ldots, Z_n$.
- **Step 5:** Compute $\underline{Y}^* = \left(\widetilde{\Sigma}_{\kappa,l}^{\epsilon}\right)^{1/2} \underline{Z}^*$, and then $\xi_t^* = \sqrt{Y_t^* + \overline{\widehat{\xi}^2}}$.
- **Step 6:** Generate the bootstrap intraday returns via

$$X_{t,n}^* = \frac{1}{\sqrt{n}} \widehat{\sigma}\left(\frac{t}{n}\right) \xi_t^*.$$

The bootstrap realized volatility is defined as:

$$RV^{LPB} := \sum_{i=1}^{n} \left(X_{i,n}^*\right)^2 = \sum_{i=1}^{n} \frac{1}{n} \widehat{\sigma}^2 \left(\frac{i}{n}\right) \left(Y_i^* + \overline{\widehat{\xi}^2}\right).$$

Compared to the asymptotic covariance $\widetilde{V}^2$ given by Theorem 3.1, we need to mimic the sum of the autocovariance of $\left(\xi_t^2\right)_{t \in \mathbb{Z}}$. Since we do not have the observations of $\{\xi_1^2, \ldots, \xi_n^2\}$, we need to estimate them via a nonparametric estimation. Therefore, certain assumptions on γ_{ξ^2} and on the banding parameter l of $\widetilde{\Sigma}_{\kappa,l}^{\epsilon}$ need to be given to ensure that, the bias of the nonparametric estimation can be neglected. These assumptions are stated as follows:

Assumption.

(B5) *Assume $|\gamma_{\xi^2}(h)| = \mathcal{O}\left(\varrho^h\right)$ for some ϱ with $|\varrho| < 1$, and $l = \lfloor a \log n \rfloor$ for some $a \in \mathbb{R}$ large enough.*

Under this assumption, the convergence rate of the positive definite autocovariance estimator $\widetilde{\Sigma}^{\epsilon}_{\kappa,l}$ is given by McMurry and Politis (2010).

Validity of the Bootstrap
Similar to McMurry and Politis (2010), we use $\rho(A) = \max_{x\in\mathbb{R}^n:|x|=1} |Ax|$ to establish convergence rates in the operator norm, where $|\cdot|$ denotes the usual Euclidean norm on $\mathbb{R}^n$. The convergence of the positive definite autocovariance estimator based on the estimated innovations is given in the following lemma.

Lemma 3.2. *Let $\{X_t : t = 1, \dots, n\}$ be given by the discrete-time model* (3.1) *and the assumptions (B1)-(B5) be fulfilled. Assume $\sup_{i\in\{0,1,\dots,n\}} \left|\widehat{\sigma}_i^2 - \sigma_i^2\right| = o_p\left(l^{-1}\right)$. It holds, as $n \to \infty$, that*

$$\rho\left(\widetilde{\Sigma}^{\epsilon}_{\kappa,l} - \Sigma_n\right) = o_p(1).$$

The validity of the bootstrap procedure is given by the following Theorem.

Theorem 3.3. *Let RV^{LPB} be estimated via the linear process bootstrap as described above. Under the same assumptions of Lemma 3.2, it holds, as $n \to \infty$, that*

$$\widetilde{T}_n^{LPB} := \sqrt{n}\left(RV^{LPB} - \frac{1}{n}\sum_{i=1}^{n}\widehat{\sigma}^2\frac{1}{n}\sum_{j=1}^{n}\widehat{\xi}_j^2\right) \xrightarrow{d} \mathcal{N}\left(0, \widetilde{V}^2\right) \tag{3.3}$$

in probability. $\widetilde{V}^2$ is defined in Theorem 3.1.
This implies as $n \to \infty$:

$$\sup_{x\in\mathbb{R}} |P(\widetilde{T}_n^{LPB} \le x) - P(\widetilde{T}_n \le x)| \xrightarrow{p} 0.$$

3.3 A simulation study

For this study, we use the same volatility function as in section 2.6, which is $\sigma(u) = 0.32(u - 0.5)^2 + 0.04$. The innovations $\{\xi_t : t = 1, \dots, n\}$ are generated by the $MA(1)$ process $\xi_t = a_1 e_{t-1} + e_t$ where e_t is i.i.d. sequence of $\mathcal{N}(0, 1/(1 + a_1^2))$, for $n = 200$ and $a_1 = 0.5$. The log-returns are simulated according to model 3.1.

Based on the log-returns, realized volatility can be computed. Then, we estimate the spot volatility via the kernel estimator given by (2.6), generate the bootstrap realizations and compute $\widetilde{T}_n^{LBB}$. The bootstrap procedure is repeated 500 times to obtain the bootstrapped quantiles. The quantiles of normal approximation are computed with estimated standard deviation $\widetilde{V}$. For $\widetilde{V}^2$, we propose the following estimator: $\underline{\widehat{\sigma}}'\widetilde{\Sigma}^{\epsilon}_{\kappa,l}\underline{\widehat{\sigma}}$, where $\underline{\widehat{\sigma}}' := (\widehat{\sigma}(1/n), \ldots, \widehat{\sigma}(1))$, $\widetilde{\Sigma}^{\epsilon}_{\kappa,l}$ is the covariance matrix estimator introduced at the beginning of section 3.2. The whole simulation is repeated 500 times to obtain boxplots of sample quantiles of interest. The true quantiles, indicated as green lines, of the finite sample distribution of $\widetilde{T}_n$ are obtained by simulation with 100.000 repetitions.

We can see in Figure 3.1, that the medians of the boxplots for normal approximation stay far away from the true quantiles, while the medians of the bootstrap boxplots are much closer, but the bootstrap quantiles vary in a wider range.

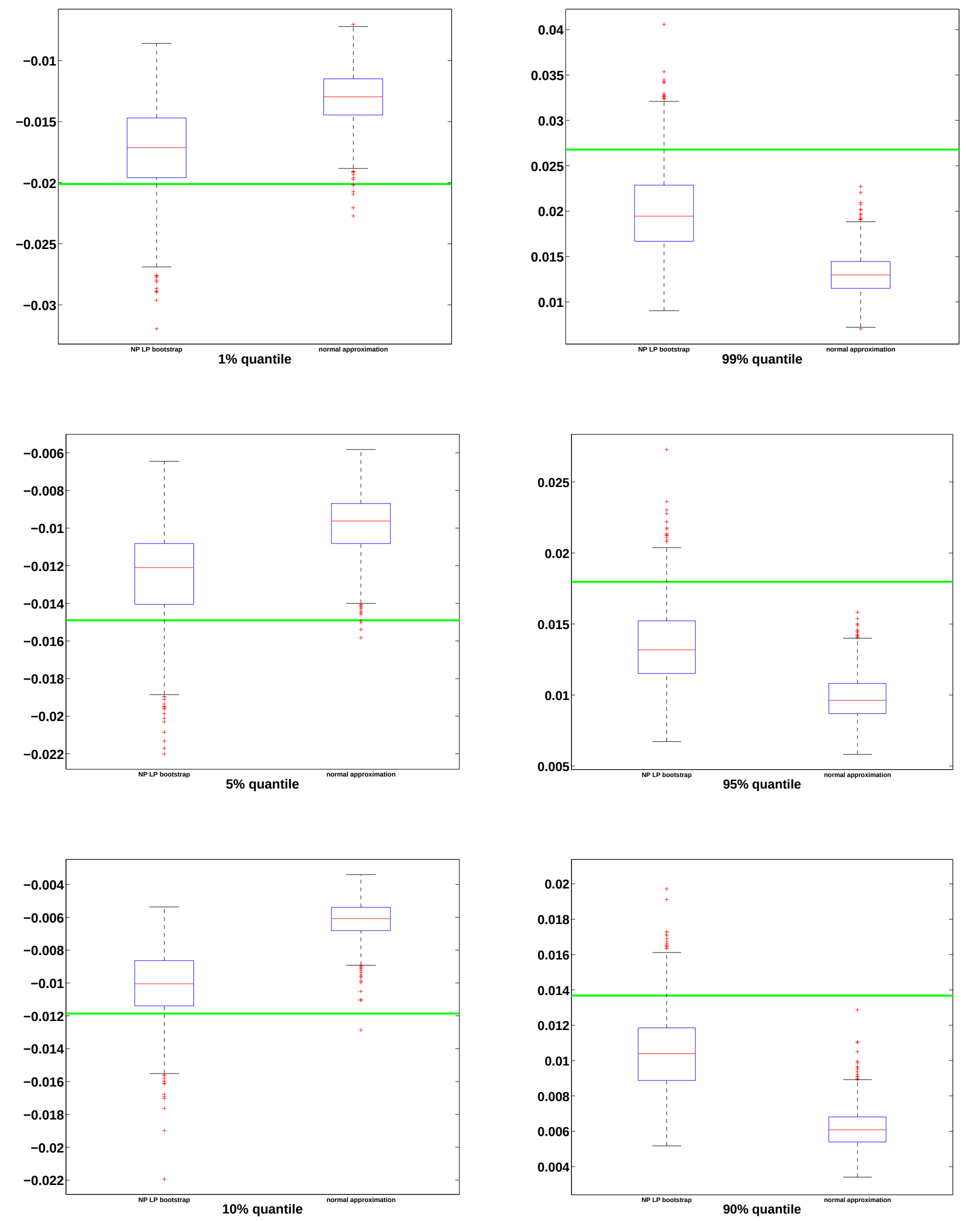

Figure 3.1: Bootstraped quantiles and quantiles of normal approximation

3.4 Proofs and auxiliary results

Proof of Theorem 3.1

To prove the limiting normal distribution (3.1) we make use of the Central Limit Theorem for triangular arrays of possibly nonstationary random variables of Neumann (2013) and Slutsky's Lemma. $\widetilde{T}_n$ can be written as follows:

$$\begin{aligned}
\widetilde{T}_n &= \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\sigma_t^2\left(\xi_t^2-1\right)+\sqrt{n}\left(\frac{1}{n}\sum_{t=1}^{n}\sigma_t^2-\int_0^1\sigma^2(u)du\right)\\
&= A_n+B_n
\end{aligned}$$

We first examine A_n.

Asymptotic variance:

The asymptotic covariance can be estimated with the following computations:

$$\begin{aligned}
VarA_n &= \frac{1}{n}\sum_{i,j=1}^{n}Cov\left(\sigma_i^2\xi_i^2,\sigma_j^2\xi_j^2\right)\\
&= \frac{1}{n}\sum_{h=-(n-1)}^{n-1}\sum_{t=1}^{n-|h|}\sigma_{t+|h|}^2\sigma_t^2\gamma_{\xi^2}(h)\\
&= \frac{1}{n}\sum_{h=-(n-1)}^{n-1}\sum_{t=1}^{n}\sigma_t^4\gamma_{\xi^2}(h)-\frac{1}{n}\sum_{h=-(n-1)}^{n-1}\sum_{t=n-|h|+1}^{n}\sigma_t^4\gamma_{\xi^2}(h)\\
&\quad+\frac{1}{n}\sum_{h=-(n-1)}^{n-1}\sum_{t=1}^{n-|h|}\sigma_t^2\mathcal{O}\left(\frac{|h|}{n}\right)\gamma_{\xi^2}(h)\\
&= A1+A2+A3.
\end{aligned}$$

Recalling $\sum_{h=-\infty}^{\infty}\gamma_{\xi^2}(h)<\infty$, and $\sigma_t<C_\sigma$ for all $t\in\{0,1,\ldots,1\}$, we have

$$A1=\frac{1}{n}\sum_{t=1}^{n}\sigma_t^4\sum_{h=-(n-1)}^{n-1}\gamma_{\xi^2}(h)\overset{n\to\infty}{\longrightarrow}\int_0^1\sigma^4(u)du\sum_{h=-\infty}^{\infty}\gamma_{\xi^2}(h):=\widetilde{V}^2,$$

and due to the Kronecker-Lemma

$$A2\leq\frac{C_\sigma}{n}\sum_{t=1}^{n}|h|\gamma_{\xi^2}(h)\overset{n\to\infty}{\longrightarrow}0.$$

In a similar fashion of $A1$ and $A2$, we have $A3\overset{n\to\infty}{\longrightarrow}0$. Due to the computations above, we have

$$Var\widetilde{T}_n=VarA_n\overset{n\to\infty}{\longrightarrow}\widetilde{V}^2$$

Lindeberg condition:

Recall that $(\xi_t)_{t=1,\ldots,n}$ and $\left(\xi_t^2\right)_{t=1,\ldots,n}$ are stationary, $E\xi_t^4<\infty$ and $\sigma_t<C_\sigma$ for all $t\in\{1,\ldots,1\}$. Let $\zeta>0$, then

$$\sum_{t=1}^{n}E\left(\frac{1}{n}\sigma_t^4\left(\xi_t^2-1\right)^2\mathbb{1}_{\left\{\left|\frac{1}{\sqrt{n}}\sigma_t^2\left(\xi_t^2-1\right)\right|>\zeta\right\}}\right)\leq\frac{C_\sigma^4}{n}\sum_{t=1}^{n}E\left(\left(\xi_t^2-1\right)^2\mathbb{1}_{\left\{\left|\xi_t^2-1\right|>\frac{\sqrt{n}\zeta}{C_\sigma^2}\right\}}\right)$$

$$= C_\sigma^4 E\left(\left(\xi_1^2-1\right)^2 \mathbb{1}_{\left\{\left|\xi_1^2-1\right|>\frac{\sqrt{n}\zeta}{C_\sigma^2}\right\}}\right) \overset{n\to\infty}{\longrightarrow} 0.$$

Weak dependance conditions:

Recall that $\xi_i = H(\ldots,\varepsilon_{i-1},\varepsilon_i)$. Let $(\varepsilon_i')_{i\in\mathbb{Z}}$ be an i.i.d. copy of $(\varepsilon_i)_{i\in\mathbb{Z}}$. Let $\widetilde{\xi}_{i,r}$ be a coupled version of ξ_i with ε_t being replaced by ε_t' for all $t \le i-r$. The geometric-moment contracting (GMC(2)) property of $(\xi_t)_{t\in\mathbb{Z}}$ leads to that there exist $C>0$ and $0<\vartheta<1$ such that for all $i\in\mathbb{N}$, $\sqrt{E\left(\xi_i-\widetilde{\xi}_{i,r}\right)^2} \le C\vartheta^r$.

Let $Y_t := \frac{1}{\sqrt{n}}\sigma_t^2\left(\xi_t^2-1\right)$, $t\in\mathbb{N}$. For all $u\in\mathbb{N}$, all indices $1\le s_1<s_2<\cdots<s_u<s_u+r=t_1\le t_2\le n$ and for all measurable square-integrable functions $g:\mathbb{R}^u\to\mathbb{R}$ with $\|g\|_\infty = \sup_{x\in\mathbb{R}^u}|g(x)|\le 1$, we have

$$\begin{aligned}
&\left|Cov\left(g\left(Y_{s_1},\ldots,Y_{s_u}\right)Y_{s_u},Y_{t_1}\right)\right| = \left|Eg\left(Y_{s_1},\ldots,Y_{s_u}\right)Y_{s_u}Y_{t_1}\right| \\
= &\left|Eg\left(Y_{s_1},\ldots,Y_{s_u}\right)\frac{1}{\sqrt{n}}\sigma_{s_u}^2\left(\xi_{s_u}^2-1\right)\frac{1}{\sqrt{n}}\sigma_{t_1}^2\left(\xi_{t_1}^2-1\right)\right| \\
= &\frac{1}{n}\sigma_{s_u}^2\sigma_{t_1}^2\left|Eg\left(Y_{s_1},\ldots,Y_{s_u}\right)\left(\xi_{s_u}^2-1\right)\left(\xi_{t_1}+\widetilde{\xi}_{t_1,r}\right)\left(\xi_{t_1}-\widetilde{\xi}_{t_1,r}\right)\right. \\
&\left.+Eg\left(Y_{s_1},\ldots,Y_{s_u}\right)\left(\xi_{s_u}^2-1\right)\left(\widetilde{\xi}_{t_1,r}^2-1\right)\right|
\end{aligned}$$

The independence of $\widetilde{\xi}_{t_1,r}$ and ξ_i, $i\le s_u$, and $E\widetilde{\xi}_j^2=1$ for all $j\in\mathbb{N}$ lead to

$$Eg\left(Y_{s_1},\ldots,Y_{s_u}\right)\left(\xi_{s_u}^2-1\right)\left(\widetilde{\xi}_{t_1,r}^2-1\right)=0.$$

We have then for $\sup_{i\in\{1,\ldots,n\}}\sigma_i<C$,

$$\begin{aligned}
&\left|Cov\left(g\left(Y_{s_1},\ldots,Y_{s_u}\right)Y_{s_u},Y_{t_1}\right)\right| \\
\le &\frac{1}{n}C^2E\left|g\left(Y_{s_1},\ldots,Y_{s_u}\right)\left(\xi_{s_u}^2-1\right)\left(\xi_{t_1}+\widetilde{\xi}_{t_1,r}\right)\left(\xi_{t_1}-\widetilde{\xi}_{t_1,r}\right)\right| \\
\le &\frac{1}{n}C^2\|g\|_\infty\sqrt{E\left(\left(\xi_{s_u}^2-1\right)\left(\xi_{t_1}+\widetilde{\xi}_{t_1,r}\right)\right)^2}\sqrt{E\left(\xi_{t_1}-\widetilde{\xi}_{t_1,r}\right)^2} \\
\le &\frac{1}{n}C^2\|g\|_\infty\sqrt[4]{E\left(\xi_{s_u}^2-1\right)^4}\sqrt[4]{E\left(\xi_{t_1}+\widetilde{\xi}_{t_1,r}\right)^4}\sqrt{E\left(\xi_{t_1}-\widetilde{\xi}_{t_1,r}\right)^2}.
\end{aligned}$$

To see the last two inequalities we make use of the Cauchy-Schwarz inequality. Since that the 8th moment of ξ_i exists, $E\left(\xi_{s_u}^2-1\right)^4<\infty$ and $E\left(\xi_{t_1}+\widetilde{\xi}_{t_1,r}\right)^4<\infty$. Together with $\|g\|_\infty\le 1$ and $GMC(2)$, we find a summable sequence $\eta_{1,r}$, i.e.

$$\eta_{1,r} := C^2\sqrt[4]{E\left(\xi_{s_u}^2-1\right)^4}\sqrt[4]{E\left(\xi_{t_1}+\widetilde{\xi}_{t_1,r}\right)^4}C\vartheta^r$$

such that

$$\left|Cov\left(g\left(Y_{s_1},\ldots,Y_{s_u}\right)Y_{s_u},Y_{t_1}\right)\right| \le n^{-1}\eta_{1,r}.$$

Analogously, we can find a summable sequence $\eta_{2,r}$, so that

$$|Cov\left(g\left(Y_{s_1},\ldots,Y_{s_u}\right),Y_{t_1}Y_{t_2}\right)| \leq n^{-1}\eta_{2,r}.$$

Let $\eta_r = max\{\eta_{1,r},\eta_{2,r}\}$. We have exactly the conditions of weak dependence needed for the Central Limit Theorem for triangular arrays of possibly nonstationary random variables of Neumann (2013), which gives

$$A_n \xrightarrow{d} \mathcal{N}\left(0,\widetilde{V}^2\right).$$

The assertion of Theorem 3.1 follows with the Lemma of Slutsky. $\square$

Proofs for the Linear process bootstrap

Recall that, Σ_n is the covariance matrix of $\underline{\xi_n^2} := \left(\xi_1^2,\ldots,\xi_n^2\right)'$; $\widehat{\Sigma}_{\kappa,l}$ is the tapered covariance matrix estimator based on the sample of $\underline{\xi_n^2}$; $\widehat{\Sigma}^{\epsilon}_{\kappa,l}$ is the positive definite version of $\widehat{\Sigma}_{\kappa,l}$.

The observations of $\xi_1^2,\ldots,\xi_n^2$ are not available, but using the nonparametric estimator $\widehat{\sigma}$ given in the section 2.3, we can estimate them. Let $\widetilde{\Sigma}_{\kappa,l}$ be the tapered covariance matrix estimator based on the estimated $\widehat{\xi}_1^2,\ldots,\widehat{\xi}_n^2$, and $\widetilde{\Sigma}^{\epsilon}_{\kappa,l}$ be the positive definite version of $\widetilde{\Sigma}_{\kappa,l}$. All the covariance matrix estimators above are introduced by McMurry and Politis (2010).

The proof is quite similar to the proof of Theorem 5 by McMurry and Politis (2010). We need only to show that under certain weak dependence conditions for $\left(\xi_i^2\right)_{i\in\mathbb{Z}}$, using $\widetilde{\Sigma}^{\epsilon}_{\kappa,l}$ instead of $\widehat{\Sigma}^{\epsilon}_{\kappa,l}$ by the bootstrap construction can mimic the variance structure as well.

Lemma 3.4. *Assume* $\sup_{i\in\{0,1,\ldots,n\}}\left|\widehat{\sigma}_i^2-\sigma_i^2\right| = \mathcal{O}_p\left(r_n\right)$, *where* $r_n\to 0$, *as* $n\to\infty$. *It holds*

$$\sup_{h\in\{0,1,\ldots,n-1\}}\left|\widehat{\gamma}_{\widehat{\xi}^2}(h)-\widehat{\gamma}_{\xi^2}(h)\right| = \mathcal{O}_p\left(r_n\right),$$

where $\{\widehat{\xi}_i^2 : i=1,\ldots,n\}$ *are estimated in Step 2 of the linear process bootstrap procedure.*

Proof. The uniform consistency of $\widehat{\sigma}_i^2$ leads to

$$\sup_{i\in\{0,1,\ldots,n\}}\left|\frac{\sigma_i^2}{\widehat{\sigma}_i^2}-1\right| = \mathcal{O}_p\left(r_n\right).$$

We have $\widehat{\xi}_i = \dfrac{\sigma_i\xi_i}{\widehat{\sigma}_i}$, $i=1,\ldots,n$. It holds

$$\sup_{i\in\{1,\ldots,n\}}\left|\widehat{\xi}_i^2-\xi_i^2\right| = \mathcal{O}_p\left(r_n\right)$$

$$\overline{\widehat{\xi}^2} = \frac{1}{n}\sum_{i=1}^{n}\frac{\sigma_i^2\xi_i^2}{\widehat{\sigma}_i^2} = \left(1+\mathcal{O}_p\left(r_n\right)\right)\frac{1}{n}\sum_{i=1}^{n}\xi_i^2.$$

It holds then uniformly in $h \in \{0, 1, \ldots, n-1\}$

$$\begin{aligned}
\widehat{\gamma}_{\widehat{\xi}^2}(h) &= \frac{1}{n}\sum_{t=1}^{n-h}\left(\widehat{\xi}_{t+h}^2 - \overline{\widehat{\xi}^2}\right)\left(\widehat{\xi}_t^2 - \overline{\widehat{\xi}^2}\right) \\
&= \frac{1}{n}\sum_{t=1}^{n-h}\left(\left(1+\mathcal{O}(r_n)\right)\xi_{t+h}^2 - \left(1+\mathcal{O}_p\left(r_n\right)\right)\frac{1}{n}\sum_{i=1}^{n}\xi_i^2\right) \\
&\quad \cdot\left(\left(1+\mathcal{O}(r_n)\right)\xi_t^2 - \left(1+\mathcal{O}_p\left(r_n\right)\right)\frac{1}{n}\sum_{i=1}^{n}\xi_i^2\right) \\
&= \left(1+\mathcal{O}(r_n)\right)\frac{1}{n}\sum_{t=1}^{n-h}\left(\xi_{t+h}^2 - \frac{1}{n}\sum_{i=1}^{n}\xi_i^2\right)\left(\xi_t^2 - \frac{1}{n}\sum_{i=1}^{n}\xi_i^2\right) \\
&= \widehat{\gamma}_{\xi^2}(h) + \mathcal{O}(r_n)
\end{aligned}$$

□

Proof of Lemma 3.2

Analogously to the proof of Theorem 1 by McMurry and Politis (2010), we have

$$\rho\left(\widetilde{\Sigma}_{\kappa,l} - \Sigma_n\right) \leq \rho\left(\widetilde{\Sigma}_{\kappa,l}^{\epsilon} - \widetilde{\Sigma}_{\kappa,l}\right) + \rho\left(\widetilde{\Sigma}_{\kappa,l} - \widehat{\Sigma}_{\kappa,l}\right) + \rho\left(\widehat{\Sigma}_{\kappa,l} - \Sigma_n\right),$$

in which

$$\begin{aligned}
\rho\left(\widetilde{\Sigma}_{\kappa,l} - \widehat{\Sigma}_{\kappa,l}\right) &\leq \max_{1\leq j\leq n}\sum_{i=1}^{n}\left|\widehat{\gamma}_{\widehat{\xi}^2}(i-j)\kappa_l\left(|i-j|\right) - \widehat{\gamma}_{\xi^2}(i-j)\kappa_l\left(|i-j|\right)\right| \\
&\leq \sum_{i=-\lfloor c_\kappa l\rfloor}^{\lfloor c_\kappa l\rfloor}\left|\widehat{\gamma}_{\widehat{\xi}^2}(i)\kappa_l(i) - \widehat{\gamma}_{\xi^2}(i)\kappa_l(i)\right| \\
&\leq (2c_\kappa l+1)\max_{i\in\{0,1,\ldots,\lfloor c_\kappa l\rfloor\}}\left|\widehat{\gamma}_{\widehat{\xi}^2}(i) - \widehat{\gamma}_{\xi^2}(i)\right| \\
&= o_p\left(1\right)
\end{aligned}$$

The final equality follows because of Lemma 3.4.
Under the conditions of Lemma 3.2, Corollary 1 of McMurry and Politis (2010) gives

$$\rho\left(\widehat{\Sigma}_{\kappa,l} - \Sigma_n\right) = o_p(1),$$

and Theorem 3 of McMurry and Politis (2010) and its proof give

$$\rho\left(\widetilde{\Sigma}_{\kappa,l}^{\epsilon} - \widetilde{\Sigma}_{\kappa,l}\right) = o_p(1).$$

The desired result is obtained.
In the proof of Lemma 2.4, it is shown that, for $l = \lfloor a \log n \rfloor$, $\sup_{i\in\{0,1,\ldots,n\}} \left|\widehat{\sigma}_i^2 - \sigma_i^2\right| = o_p\left(l^{-1}\right)$ holds true.

Proof of Theorem 3.3
We consider the asymptotic behavior of

$$\widetilde{T}_n^{LPB} = \sqrt{n}\left(RV^{LPB} - \frac{1}{n}\sum_{i=1}^n \widehat{\sigma}^2 \frac{1}{n}\sum_{j=1}^n \widehat{\xi}_j^2\right) = \sum_{i=1}^n \frac{1}{\sqrt{n}}\widehat{\sigma}_i^2 Y_i^*.$$

$$\begin{aligned} E^* RV^{LPB} &= \sum_{i=1}^n \frac{1}{n}\widehat{\sigma}_i^2 E^*\left(Y_i^* + \overline{\widehat{\xi}^2}\right) \\ &= \sum_{i=1}^n \frac{1}{n}\widehat{\sigma}_i^2 E^*\left(\left(\widetilde{\Sigma}_{\kappa,l}^{\epsilon}\right)^{1/2}\underline{Z}^*\right) + \sum_{i=1}^n \frac{1}{n}\widehat{\sigma}_i^2\overline{\widehat{\xi}^2} = \frac{1}{n}\sum_{i=1}^n \widehat{\sigma}_i^2\overline{\widehat{\xi}^2}. \end{aligned}$$

The standardizing in the procedure step 3 leads to the last equality above.
The computation of its asymptotic variance is given as follows.
Asymptotic variance:
Let $\underline{\widehat{\sigma}^2} = \left(\widehat{\sigma}_1^2, \ldots, \widehat{\sigma}_n^2\right)'$. Due to Lemma 3.2, Lemma 2 of McMurry and Politis (2010) and its proof, we have

$$n^{-\frac{1}{2}}\underline{\widehat{\sigma}^2}'\underline{Y}^* = n^{-\frac{1}{2}}\underline{\widehat{\sigma}^2}'\left(\widetilde{\Sigma}_{\kappa,l}^{\epsilon}\right)^{1/2}\underline{Z}^* = n^{-\frac{1}{2}}\underline{\widehat{\sigma}^2}'\left(\Sigma_n\right)^{1/2}\underline{\widetilde{Z}}^* + o_p(1),$$

where $(\Sigma_n)^{1/2}$ is the lower-triangular matrix associated with the Cholesky decomposition of $\Sigma_{n,k}$, $\underline{\widetilde{Z}}^*$ is the equivalent bootstrap resample to Z^*, except the resample is drawn form the standardized values of $(\Sigma_n)^{1/2}\underline{Y}$.
Since $E^*\left(\underline{\widetilde{Z}}^*\underline{\widetilde{Z}}^{*\prime}\right) = I$ (I denotes Identity matrix), we have

$$\begin{aligned} Var^*\widetilde{T}_n^{LPB} &= Var^*\left(n^{-\frac{1}{2}}\underline{\widehat{\sigma}^2}'\underline{Y}^*\right) \\ &= Var^*\left(n^{-\frac{1}{2}}\underline{\widehat{\sigma}^2}'\left(\Sigma_n\right)^{1/2}\underline{\widetilde{Z}}^*\right) + o_p(1) \\ &= \frac{1}{n}E^*\left(\underline{\widehat{\sigma}^2}'\left(\Sigma_n\right)^{1/2}\underline{\widetilde{Z}}^*\underline{\widetilde{Z}}^{*\prime}\left(\Sigma_n\right)^{-1/2}\underline{\widehat{\sigma}^2}\right) + o_p(1) \\ &= \frac{1}{n}\sum_{h=-(n-1)}^{n-1}\sum_{t=1}^{n-|h|}\widehat{\sigma}_t^2\widehat{\sigma}_{t+h}^2\gamma_{\xi^2}(h) + o_p(1) \\ &= \frac{1}{n}\sum_{h=-(n-1)}^{n-1}\sum_{t=1}^{n}\widehat{\sigma}_t^2\widehat{\sigma}_{t+h}^2\gamma_{\xi^2}(h) - \frac{1}{n}\sum_{h=-(n-1)}^{n-1}\sum_{t=n-|h|+1}^{n}\widehat{\sigma}_t^2\widehat{\sigma}_{t+h}^2\gamma_{\xi^2}(h) + o_p(1) \\ &= A1 + A2 + o_p(1). \end{aligned}$$

We consider the first term and obtain

$$\begin{aligned}
A1 &= \frac{1}{n}\sum_{t=1}^{n}\sigma_t^4 \sum_{h=-(n-1)}^{n-1}\gamma_{\xi^2}(h) + \frac{1}{n}\sum_{t=1}^{n}\left(\widehat{\sigma}_t^4 - \sigma_t^4\right)\sum_{h=-(n-1)}^{n-1}\gamma_{\xi^2}(h) \\
&\quad + \frac{1}{n}\sum_{h=-(n-1)}^{n-1}\sum_{t=1}^{n}\widehat{\sigma}_t^2\left(\widehat{\sigma}_{t-h}^2 - \widehat{\sigma}_t^2\right)\gamma_{\xi^2}(h) \\
&= B1 + B2 + B3.
\end{aligned}$$

where, as $n \to \infty$,

$$B1 \longrightarrow \int_0^1 \sigma^4(u)du \sum_{h=-\infty}^{\infty}\gamma_{\xi^2}(h).$$

Since $\sum_{h=-\infty}^{\infty}\gamma_{\xi^2}(h) < \infty$, due to the proof of Lemma 2.4,

$$B2 \leq o_p(1)\sum_{h=-\infty}^{\infty}\gamma_{\xi^2}(h) = o_p(1).$$

Similarly to the computation (2.9) in the proof of Lemma 2.4, it can be shown

$$\sup_{t\in\{h,\dots,n\}}\left|\widehat{\sigma}_t^2 - \widehat{\sigma}_{t-h}^2\right| = \mathcal{O}_p\left(\frac{h\log(n)}{n\widetilde{h}}\right),$$

where $\widetilde{h}$ is the bandwidth in the kernel estimator. We have then

$$\begin{aligned}
B3 &= \frac{1}{n}\sum_{h=-(n-1)}^{n-1}\sum_{t=1}^{n}\widehat{\sigma}_t^2\mathcal{O}_p\left(\frac{h\log(n)}{n\widetilde{h}}\right)\gamma_{\xi^2}(h) \\
&\leq \sup_{t\in\{1,\dots,n\}}\widehat{\sigma}_t^2\mathcal{O}_p\left(\frac{\log(n)}{n\widetilde{h}}\right)\sum_{h=-(n-1)}^{n-1}|h|\gamma_{\xi^2}(h).
\end{aligned}$$

The uniform consistency of $\widehat{\sigma}^2$ leads to $\widehat{\sigma}_t^2 < \infty$ in probability for all $t \in \{1,\dots,n\}$ and the assumption *(B5)* leads to $\sum_{h=-\infty}^{\infty} h\gamma_{\xi^2}(h) < \infty$. For a bandwidth parameter $\widetilde{h}$ with $\frac{\log(n)}{n\widetilde{h}} \to 0$ as $n \to \infty$, we have $B3 = o_p(1)$.
The uniform consistency of $\widehat{\sigma}^2$ leads to $\widehat{\sigma}_t^2\widehat{\sigma}_{t+h}^2 < \infty$ in probability for all t and h. Due to the Kronecker-Lemma, it holds $A2 = o_p(1)$. We have therefore

$$Var^*\widetilde{T}_n^{LPB} \xrightarrow{p} \widetilde{V}^2.$$

The further proof is similar to the one of Theorem 5 in McMurry and Politis (2010). They showed $\rho\left(L_{n,k} - (\Sigma_n)^{1/2}\right) \to 0$, where $L_{n,k}$ is the lower-triangular matrix associated with the Cholesky decomposition of $\Sigma_{n,k}$. Using $L_{n,k}$ to approximate $(\Sigma_n)^{1/2}$,

$n^{-\frac{1}{2}}\underline{\widehat{\sigma}^2}'\underline{Y}^*$ can be approximated with $n^{-\frac{1}{2}}\underline{\widehat{\sigma}^2}'\left(L_{n,k}\right)\widetilde{Z}^*$, which can be described as the sum of linear combination of independent variables $\{\widetilde{Z}_1^*, \ldots, \widetilde{Z}_n^*\}$. The Central Limit Theorem for triangular arrays yields the desired result. $\square$

4 Bootstrapping realized covariance

This chapter deals with a covolatility estimator, named realized covariance, which is the multivariate version of the realized volatility estimator. Based on high dimensional discrete time models, standard first-order asymptotic theories for the integrated covariance are given. Considering the model with independent innovations, in which the intraday returns in a local area are nearly i.i.d., we propose in section 4.2 a local bootstrap procedure by resampling the neighboring intraday returns. For the model with weakly dependent innovations, we propose the local block bootstrap and the local wild dependent bootstrap to mimic the dependent structure, and therefore approximate the distribution of the realized covariance.

4.1 Introduction

Similarly to chapter 2 with univariate models, we introduce first some definitions and results based on multivariate continuous-time models. Consider d financial asset over a day. A standard multivariate continuous-time model for the log-price process is given by

$$\underline{P}_t = \underline{\alpha}_t + \int_0^t \Theta(u) d\underline{W}(u)$$

where $\underline{\alpha}$ denotes a drift process of the vector of assets, Θ ist the **spot covolatility process**, and $\underline{W}$ is a vector standard Brownian motion. The **spot covariance** at time u is defined as $\Sigma(u) = \Theta(u)\Theta(u)'$ with elements $\{\Sigma_{kl}(u) : k, l = 1, \dots, d\}$.

We assume that equidistant high frequency intraday data with lag $1/n, n \in \mathbb{N}$ is observable. $\underline{X}_i := \underline{P}_{\frac{i}{n}} - \underline{P}_{\frac{i-1}{n}}$ denotes the vector of the intraday log-returns over the time interval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$.

The **integrated covariance matrix (ICV)** over a day is defined as

$$ICV := \int_0^1 \Sigma(u)du,$$

and a consistent estimator of ICV, named **realized covariance (RCV)**, is defined as

$$RCV_n := \sum_{i=1}^{n} \underline{X}_i \underline{X}_i'.$$

The asymptotic law of $\sqrt{n}\,(RCV_n - ICV)$ is under some assumptions mixed normal with mean 0 and a random covariance matrix. (see Barndorff-Nielsen and Shephard (2004b)). The result, conditioning on the path of Θ, for the component (k,l) of the realized covariance as $n \to \infty$ is that

$$\frac{\sqrt{n}\left(\sum_{i=1}^{n} X_{(k)i}X_{(l)i} - \int_0^1 \Sigma_{kl}(u)du\right)}{\rho_{kl}} \xrightarrow{d} \mathcal{N}\left(0,1\right),$$

$k,l = 1,\cdots,d$, where $X_{(k)i}$ denotes the kth element of the return $\underline{X}_i$, Σ_{kl} denotes the (k,l)th element of the spot covariance Σ, and

$$\rho_{kl}^2 = \int_0^1 \left(\Sigma_{kk}(u)\Sigma_{ll}(u) + \Sigma_{kl}^2(u)\right) du. \tag{4.1}$$

A consistent estimator of ρ_{kl}^2 (see also Barndorff-Nielsen and Shephard (2004b)) is given as follows

$$\widehat{\rho_{kl}^2} = n\left(\sum_{i=1}^{n} X_{(k)i}^2 X_{(l)i}^2 - \sum_{j=1}^{n-1} X_{(k)j}X_{(l)j}X_{(k)j+1}X_{(l)j+1}\right). \tag{4.2}$$

As an alternative tool to the standard normal approximation, Dovonon, Gonçalves and Meddahi (2013) introduced an i.i.d Bootstrap procedure for a multivariate continuous semimartingale model, which consists of resampling the vectors of log-returns $\underline{X}_i$ in an i.i.d. scheme from the set $\{\underline{X}_i : i = 1,\cdots,n\}$.

The nonparametric bootstrap procedures, that we proposed for the univariate models are not applicable, since we can estimate Σ via a kernel estimator, but $\Sigma = \Theta\Theta'$ does not identify Θ without knowing the precise structure of Θ, and therefore we cannot estimate the innovations correctly. Instead, we propose the local bootstrap for the multivariate model with independent innovations, and the local block bootstrap and the local wild bootstrap for the multivariate model with weakly dependent innovations in the following sections.

4.2 Multivariate model with independent innovations

In this section, we consider a relative simple situation, namely a multivariate discrete time model with independent setup, to get knowing of how a local resampling mechanism works.

4.2.1 Model and the assumptions

We consider the following discrete-time model for the multivariate intraday log-return process:

$$\underline{X}_t = \frac{1}{\sqrt{n}}\Theta\left(\frac{t}{n}\right)\underline{\varepsilon}_t, \quad t = 1, \ldots, n. \tag{4.3}$$

We will use the index $k = 1, 2, \ldots, d$ to denote the kth asset, e.g. $X_{(k)t}$ (the kth elements of $\underline{X}_t$) denotes the log-price of the kth asset at the time $\frac{t}{n}$.

Assumption.
(C1) *Θ denotes a $d \times c$ dimensional spot covolatility term. Its elements $\{\theta_{ki} : k = 1, \ldots, d; i = 1, \ldots, c\}$ can be described by non-stochastic continuous differentiable functions $\theta_{ki} : [0, 1] \to (0, \infty)$ with a first derivative which is bounded from above.*
(C2) *$\underline{\varepsilon}_t$ denotes a c-dimensional vector of noise innovations, which are i.i.d. in $t = 1, \ldots, n$ with elements $\{\varepsilon_{i,t} : i = 1, \cdots, c\}$, which are independent but not necessarily normally distributed with $E\varepsilon_{1,t} = 0$, $E\varepsilon_{1,t}^2 = 1$, $E\varepsilon_{1,t}^4 = \kappa < \infty$ and $E|\varepsilon_{1,t}|^{8+\Delta} < \infty$ for some $\Delta > 0$.*

We introduce first two operators *vec* and *vech*. Let $\underline{M}$ be a $d \times d$ dimensional matrix. $vec(\underline{M})$ stacks the columns of the matrix $\underline{M}$ into a vector (see e.g. Lutkepohl (2006)), is therefore a vector with d^2 elements. $vech(\underline{M})$ stacks the lower triangular elements of the columns of the matrix $\underline{M}$ into a vector with $\frac{(d+1)d}{2}$ elements. In this thesis, we make use of *vec* to present our results, which imply similar results by using *vech*. A Central Limit Theorem for $\sum_{t=1}^{n} vec\,(\underline{X}_t \underline{X}_t')$ based on the multivariate discrete-time model is given as follows.

Theorem 4.1. *For the discrete-time model* (4.3)*, it holds under assumptions (C1) and (C2), as $n \to \infty$, that*

$$\underline{T}_n := \sqrt{n}\left(vec\left(\sum_{t=1}^{n} \underline{X}_t \underline{X}_t'\right) - vec\left(\int_0^1 \Sigma(u)du\right)\right) \xrightarrow{d} \mathcal{N}\left(0, \underline{V}\right),$$

where $\underline{V}$ *is an* $d^2 \times d^2$ *matrix with elements*

$$V_{klk'l'} := \int_0^1 \left(\sum_{i=1}^{c} (\kappa - 3)\theta_{ki}(u)\theta_{li}(u)\theta_{k'i}(u)\theta_{l'i}(u) + \Sigma_{kk'}(u)\Sigma_{ll'}(u) + \Sigma_{kl'}(u)\Sigma_{k'l}(u) \right) du,$$

$k, l, k', l' = 1, \ldots, d.$

Remark 4.2. *Considering the component* $X_{(k),t}X_{(l),t}$ *of the vector* $vec\,(\underline{X}_t\underline{X}_t)$*, we have a Central Limit Theorem as follows:*
It holds under the same assumptions of Theorem 4.1, as $n \to \infty$*, that*

$$T_{kl,n} := \sqrt{n} \left(\sum_{t=1}^{n} X_{(k),t}X_{(l),t} - \int_0^1 \Sigma_{kl}(u)du \right) \xrightarrow{d} \mathcal{N}\left(0, V_{kl}\right),$$

where

$$V_{kl} := \int_0^1 \left(\sum_{i=1}^{c} (\kappa - 3)\theta_{ki}^2(u)\theta_{li}^2(u) + \Sigma_{kk}(u)\Sigma_{ll}(u) + \Sigma_{kl}^2(u) \right) du,$$

$k, l = 1, \ldots, d.$
Compared to the asymptotic covariance given by (4.1)*, the forth order moment appears here, while in* (4.1) *disappears. To obtain a confidence interval of* $T_{kl,n}$*, we need to estimate* V_{kl}*. The one given by* (4.2) *is a consistent estimator of* V_{kl} *as well.*

Now we pose the following question: Are the nonparametric bootstrap algorithms, we proposed in capital 2, still applicable?
We observe that, the asymptotic covariance $\underline{V}$ in Theorem 4.1 depends not only on the structure of Σ but also on the structure of Θ. Using the kernel estimator given by (2.6) in section 2.3, we could estimate Σ, and under the assumption that $\kappa = 3$, e.g. $\varepsilon_{i,t} \sim \mathcal{N}(0, 1)$, we could mimic the covariance correctly without knowing the precise structure of Θ. Then, we could use for example the Cholesky decomposition $\Sigma = \widetilde{\Theta}\widetilde{\Theta}'$ to estimate an $\widetilde{\Theta}$ (not necessarily consistent to Θ) and apply the nonparametric bootstrap algorithms.
Since the decomposition is not unique, we could not identify the real precise structure of Θ. That means, without the assumption $\kappa = 3$, the nonparametric i.i.d bootstrap and nonparametric wild bootstrap are not applicable.

4.2.2 The local bootstrap

The high dimensional intraday log-returns given by the model (4.3) are independently but not identically distributed. Shi (1991) introduced the local bootstrap method based on a univariate kernel regression model. This bootstrap method is useful for the heteroscedastic data. We apply the local bootstrap procedure by resampling the vectors of discrete time log-returns from a local interval of each data point, where the log-returns are independently and nearly identically distributed.

Bootstrap procedure

Suppose that observations of the log-returns $\underline{X}_1, \cdots, \underline{X}_n$ are available.

- **Step 1:** Because of the boundary effect by the resampling at the first and last observations, we suppose to add some data point at the left side (see figure 4.1) and the right side (see figure 4.2) of the original data set.

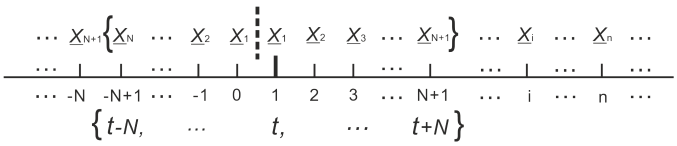

Figure 4.1: Resampling of the first observations

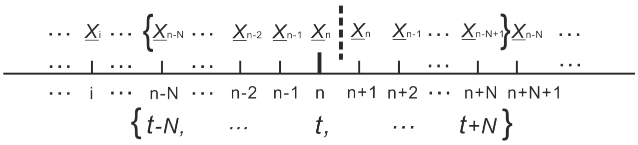

Figure 4.2: Resampling of the last observations

- **Step 2:** Select a number $N \in \mathbb{N}$ with $N \to \infty$ and $\dfrac{N}{n} \to 0$, as $n \to \infty$. $2N+1$ is the size of the local window. Generate the bootstrap intraday returns $\{\underline{X}_t^* : t = 1, \cdots, n\}$ via $\underline{X}_t^* = \underline{X}_{T_t^*}$, in which $T_t^* \sim$ Laplace on the set $\{t-N, \cdots, t+N\}$.

- **Step 3:** The bootstrap realized covariation is estimated via

$$RCV_n^* := \sum_{t=1}^{n} \underline{X}_t^* \underline{X}_t^{*\prime}.$$

Step 1 guarantee that, the observations in the boundary area will be drawn with the same probability as the observations in the middle. The bootstrap expectation value $E^* RCV_n^*$ can be therefore explicitly given as a $d \times d$ matrix with elements:

$$E^* RCV_n^* \quad = \quad \left\{ \sum_{t=1}^{n} E^* X_{(k)t}^* X_{(l)t}^* \right\}_{l,k=1,\dots,d} = \left\{ \sum_{t=1}^{n} \sum_{j=t-N}^{t+N} \frac{1}{2N+1} X_{(k)j} X_{(l)j} \right\}_{l,k=1,\dots,d}$$

$$= \left\{\sum_{t=1}^{n} X_{(k)t}X_{(l)t}\right\}_{l,k=1,\dots,d} = \sum_{t=1}^{n} \underline{X}_t\underline{X}_t'. \tag{4.4}$$

Validity of the Bootstrap

The convergence rates of the bootstrap expectation values are given as follows:

Lemma 4.3. *Let $\{\underline{X}_t : t = 1, \dots, n\}$ be given by the discrete-time model 4.3 and the assumptions (C1) and (C2) be fulfilled. Let $\{\underline{X}_t^* : t = 1, \dots, n\}$ be estimated via the local bootstrap as described above. It holds true for any $k, l, k', l' = 1, \dots, d$, that*

$$\sup_{t\in\{1,\dots,n\}} \left\{E^*\left(X^*_{(k)t}X^*_{(l)t}\right) - E\left(X_{(k)t}X_{(l)t}\right)\right\} = \mathcal{O}_p\left(\frac{N}{n^2}\right)$$

$$\sup_{t\in\{1,\dots,n\}} \left\{E^*\left(X^*_{(k)t}X^*_{(l)t}X^*_{(k')t}X^*_{(l')t}\right) - E\left(X_{(k)t}X_{(l)t}X_{(k')t}X_{(l')t}\right)\right\} = \mathcal{O}_p\left(\frac{N}{n^3}\right).$$

The validity of the bootstrap procedure is given by the following Theorem:

Theorem 4.4. *Let $\{\underline{X}_t : t = 1, \dots, n\}$ be given by the discrete-time model 4.3 and RCV_n^* be estimated via the local bootstrap. Under the same assumptions of Lemma 4.3, it holds true, as $n \to \infty$, that*

$$\underline{T}_n^{LB} := \sqrt{n}\left(vec\left(RCV_n^*\right) - vec\left(E^*RCV_n^*\right)\right) \xrightarrow{d} \mathcal{N}\left(0, \underline{V}\right),$$

in probability, where E^*RCV^* is given by (4.4) and $\underline{V}$ is defined in Theorem 4.1. The result implies the validity of the local bootstrap procedure:

$$\sup_{x\in\mathbb{R}^{d^2}} |P(\underline{T}_n^{LB} \le x) - P(\underline{T}_n \le x)| \xrightarrow{p} 0.$$

4.2.3 A simulation study

Analogously to the simulation study in chapter 2, we compare the accuracy of the proposed bootstrap method with the normal approximation by considering 2.5% and 97.5% quantiles of the components of $\underline{T}_n$ (see Remark 4.2).

Assume that, the observations are simulated according to the following bivariate model:

$$\underline{X}_t = \frac{1}{\sqrt{n}}\Theta\left(\frac{t}{n}\right)\underline{\varepsilon}_t, \quad t = 1, \dots, n,$$

where $\Theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$ and $\underline{\varepsilon}_t = (\varepsilon_{1,t}, \varepsilon_{2,t})'$.

We choose here the sample size $n = 100$ and the spot volatility functions as follows:

$\theta_{11}(u) = 0.64 * (u - 0.5)^2 + 0.08$, $\theta_{12}(u) = 0.24 - 0.64 * (u - 0.5)^2$, $\theta_{21}(u) = 0.16 + 0.08\sin(2\pi u)$, and a constant function $\theta_{22}(u) = 0.16$. The spot covariance $\Sigma = \Theta\Theta'$ is visualized in Figure 4.3.

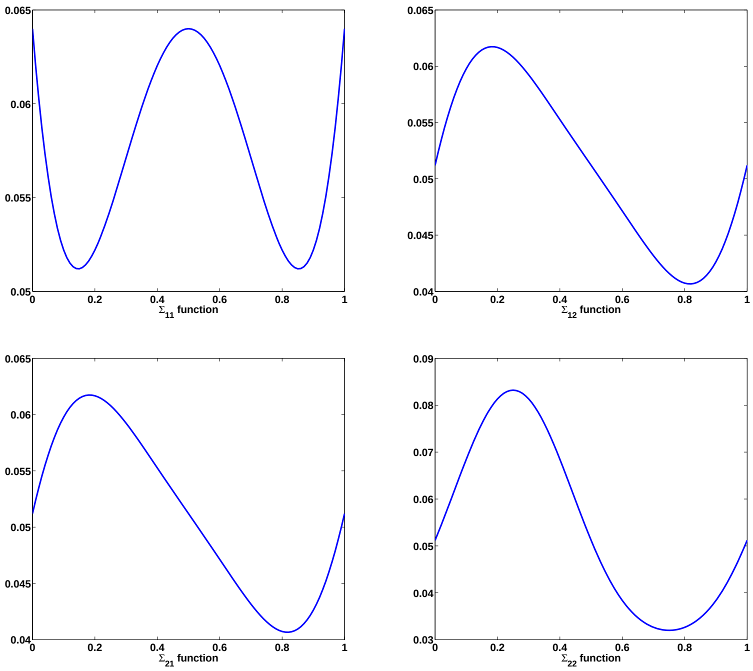

Figure 4.3: Spot covariance functions

On the one hand, we generate the bootstrap data due to the local bootstrap procedures and compute $\underline{T}_n^{LB}$. With 1000 repetitions of the bootstrap procedures, we get empirical quantiles of the distribution of the components from $\underline{T}_n^{LB}$. On the other hand, we computed the desired quantiles via normal approximation with estimated covariance (see Remark 4.2). The whole simulation again is repeated 1000 times to obtain boxplots of sample quantiles of interest.

The finite sample properties of $T_{12,n} := \sqrt{n}\left(\sum_{t=1}^{n} X_{(1)t}X_{(2)t} - \int_0^1 \Sigma_{12}(u)du\right)$, where $\Sigma_{12}(u) = \theta_{11}\theta_{21} + \theta_{12}\theta_{22}$, is shown in Figure 4.4. The left panel is for the model with i.i.d. $\varepsilon_{i,t} \sim N(0,1)$, $i = 1,2$ and $t = 1,\ldots,n$, while the right one is for the model with i.i.d. $\varepsilon_{i,t} \sim exp(1) - 1$, $i = 1,2$ and $t = 1,\ldots,n$. The boxplots on the left side of each panel give the approximations via local bootstrap, while the ones on the right side of each panel give the results obtained from normal approximation. The true quantiles, indicated as green lines, of the finite sample distribution of $T_{12,n}$ are obtained by simulation with 100.000 repetitions.

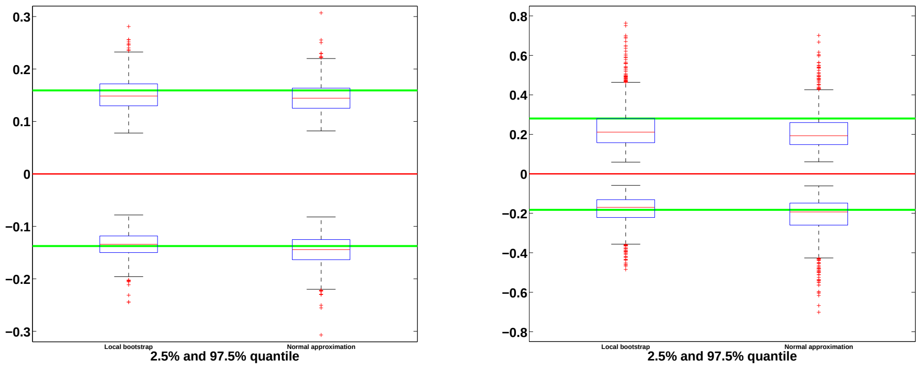

Figure 4.4: Quantiles of local bootstrap and normal approximation

One can see that the true finite sample distribution is skew. The medians of both bootstrap boxplots nearly hit the true 2.5% quantile. For the 97.5% quantile, the local bootstrap performs not as well as for the 2.5% quantile in both panels, but at least slightly better than the normal approximation.

4.2.4 Proofs and auxiliary results

To simplify the notation, we use $\theta_{ki,t}$ instead of $\theta_{ki}\left(\frac{t}{n}\right)$ in the following proofs.

Proof of Theorem 4.1

The Central Limit Theorem for the triangular array of the sum of independent, non-identically distributed random variables, Cramér-Wold Device and Lemma of Slutsky lead to a multivariate asymptotic normality of $\underline{T}_n$ from Theorem 4.1. We only show here the computation of the asymptotic covariance matrix. Recalling that $\{\varepsilon_{i,t} : i = 1, \ldots, c; t = 1 \ldots, n\}$ are independent with $E\varepsilon_{i,t} = 0$, $E\varepsilon_{i,t}^2 = 1$, $E\varepsilon_{i,t}^4 = \kappa < \infty$, we have for any $k, k', l, l' \in \{1, \ldots, d\}$ that

$$
\begin{aligned}
& Cov\left(\sqrt{n}\sum_{t=1}^{n} X_{(k)t}X_{(l)t}, \sqrt{n}\sum_{t=1}^{n} X_{(k')t}X_{(l')t}\right) \\
= \; & n\sum_{t=1}^{n}\left(E\left(X_{(k)t}X_{(l)t}X_{(k')t}X_{(l')t}\right) - E\left(X_{(k)t}X_{(l)t}\right)E\left(X_{(k')t}X_{(l')t}\right)\right) \\
= \; & \frac{1}{n}\sum_{t=1}^{n}\left(\sum_{i,j,p,q=1}^{c} \theta_{ki,t}\theta_{lj,t}\theta_{k'p,t}\theta_{l'q,t}E\left(\varepsilon_{i,t}\varepsilon_{j,t}\varepsilon_{p,t}\varepsilon_{q,t}\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{i,j=1}^{c} \theta_{ki,t}\theta_{lj,t} E\left(\varepsilon_{i,t}\varepsilon_{j,t}\right) \sum_{p,q=1}^{c} \theta_{k'p,t}\theta_{l'q,t} E\left(\varepsilon_{p,t}\varepsilon_{q,t}\right)\Bigg) \\
= \; & \frac{1}{n}\sum_{t=1}^{n}\Bigg(\sum_{i=1}^{c}\theta_{ki,t}\theta_{li,t}\theta_{k'i,t}\theta_{l'i,t}\kappa + \sum_{i,p=1,i\neq p}^{c} \theta_{ki,t}\theta_{li,t}\theta_{k'p,t}\theta_{l'p,t} \\
&+ \sum_{i,j=1,i\neq j}^{c} \theta_{ki,t}\theta_{lj,t}\theta_{k'i,t}\theta_{l'j,t} + \sum_{i,j=1,i\neq j}^{c} \theta_{ki,t}\theta_{lj,t}\theta_{k'j,t}\theta_{l'i,t} - \sum_{i=1}^{c}\theta_{ki,t}\theta_{li,t}\sum_{j=1}^{c}\theta_{k'j,t}\theta_{l'j,t}\Bigg) \\
= \; & \frac{1}{n}\sum_{t=1}^{n}\left(\sum_{i=1}^{c}(\kappa-3)\theta_{ki,t}\theta_{li,t}\theta_{k'i,t}\theta_{l'i,t} + \Sigma_{kk',t}\Sigma_{ll',t} + \Sigma_{kl',t}\Sigma_{k'l,t}\right) \stackrel{n\to\infty}{\longrightarrow} V_{klk'l'},
\end{aligned}
$$

which is the elements of the asymptotic covariance matrix given in Theorem 4.1. □

Proofs for the local bootstrap

Proof of Lemma 4.3

We will show here, that the second and fourth order Properties are correctly imitated through the local bootstrap algorithm. These properties will be applied to the proof of the validity of the bootstrap algorithm later.

Note that $\{\theta_{kl}(u) : k, l = 1, \ldots, d\}$ are non-stochastic continuous differentiable functions with bounded $\theta'_{kl}(u)$, $u \in [0,1]$ (say the upper bound is $\widetilde{\theta}'$). It holds uniformly in t that

$$
\sup_{i\in\{t-N,\ldots,t+N\}} |\theta_{kp,i}\theta_{lq,i} - \theta_{kp,t}\theta_{lq,t}| = \frac{N}{n}\widetilde{\theta}'\theta_{lq,i} + \frac{N}{n}\widetilde{\theta}'\theta_{kp,i} + \left(\frac{N}{n}\widetilde{\theta}'\right)^2 = \mathcal{O}\left(\frac{N}{n}\right). \quad (4.5)
$$

According to the local bootstrap procedure, that $\{X^*_{(k)i} : i = t-N, \ldots, t+N\}$ are independently identically uniformly distributed, it holds uniformly in t

$$
\begin{aligned}
E\left[E^*\left(X^*_{(k)t}X^*_{(l)t}\right)\right] &= E\left[\frac{1}{2N+1}\sum_{i=t-N}^{t+N} X_{(k)i}X_{(l)i}\right] \\
&= \frac{1}{2N+1}\sum_{i=t-N}^{t+N}\frac{1}{n}\sum_{p,q=1}^{c}\theta_{kp,i}\theta_{lq,i}E\left(\varepsilon_{p,i}\varepsilon_{q,i}\right) \\
&\stackrel{\triangle}{=} \frac{1}{2N+1}\frac{1}{n}\sum_{i=t-N}^{t+N}\sum_{p,q=1}^{c}\theta_{kp,t}\theta_{lq,t}E\left(\varepsilon_{p,1}\varepsilon_{q,1}\right) + \mathcal{O}\left(\frac{N}{n^2}\right) \\
&= \frac{1}{n}\sum_{p,q=1}^{c}\theta_{kp,t}\theta_{lq,t}E\left(\varepsilon_{p,1}\varepsilon_{q,1}\right) + \mathcal{O}\left(\frac{N}{n^2}\right) \\
&= E\left(X_{(k)t}X_{(l)t}\right) + \mathcal{O}\left(\frac{N}{n^2}\right)
\end{aligned}
$$

To see equality $\triangle$ above, (4.5) is used. Since $E\varepsilon_{p,i}^2 < \infty$ and $E\varepsilon_{p,i}^4 < \infty$ for any $p \in \{1, \dots, c\}$ and $i \in \{1, \dots, n\}$, it holds as $n \to \infty$, that

$$Var\left[E^*\left(X^*_{(k)t}X^*_{(l)t}\right)\right] = \left(\frac{1}{2N+1}\right)^2 \sum_{i=t-N}^{t+N} \frac{1}{n^2} Var\left(\sum_{p,q=1}^{c} \theta_{kp,i}\theta_{lq,i}\varepsilon_{p,i}\varepsilon_{q,i}\right) \longrightarrow 0$$

This leads to the following uniform consistency in t:

$$E^*\left(X^*_{(k)t}X^*_{(l)t}\right) = E\left(X_{(k)t}X_{(l)t}\right) + \mathcal{O}_p\left(\frac{N}{n^2}\right). \tag{4.6}$$

Analogously, it holds uniformly in t:

$$E^*\left(X^*_{(k)t}X^*_{(l)t}X^*_{(k')t}X^*_{(l')t}\right) = E\left(X_{(k)t}X_{(l)t}X_{(k')t}X_{(l')t}\right) + \mathcal{O}_p\left(\frac{N}{n^3}\right). \tag{4.7}$$

in which $E\left|\varepsilon_{i,t}\right|^{8+\Delta} < \infty$ is required. $\square$

Proof of Theorem 4.4

Recalling the local bootstrap procedure, given the observations $\{\underline{X}_t : t = 1, \dots, n\}$, $\{\underline{X}_t^*\underline{X}_t^{*\prime} : t = 1, \dots, n\}$ are independent. The CLT for independent triangular arrays can be applied. The proof of Theorem 4.4 will be given in two steps. We begin with the convergence of the bootstrap covariances of the statistics $\underline{T}_n^{LB}$ with the following elements

$$\sqrt{n}\left(\sum_{t=1}^{n} X^*_{(k)t}X^*_{(l)t} - \sum_{i=1}^{n} X_{(k)i}X_{(l)i}\right), \quad k, l = 1, \dots, d,$$

and then prove the multivariate asymptotic normality.

Asymptotic covariance:

Applying (4.6) and (4.7), we obtain

$$\begin{aligned}
& Cov^*\left(\sum_{t=1}^{n} X^*_{(k)t}X^*_{(l)t} - \sum_{i=1}^{n} X_{(k)i}X_{(l)i}, \sum_{t=1}^{n} X^*_{(k')t}X^*_{(l')t} - \sum_{i=1}^{n} X_{(k')i}X_{(l')i}\right) \\
= & \sum_{t=1}^{n} Cov^*\left(X^*_{(k)t}X^*_{(l)t}, X^*_{(k')t}X_{(l')t}\right) \\
= & \sum_{t=1}^{n} E^*\left(X^*_{(k)t}X^*_{(l)t}X^*_{(k')t}X^*_{(l')t}\right) - \sum_{t=1}^{n} E^*\left(X^*_{(k)t}X^*_{(l)t}\right) E^*\left(X^*_{(k')t}X^*_{(l')t}\right) \\
= & \sum_{t=1}^{n} \left(E\left(X_{(k)t}X_{(l)t}X_{(k)t}X_{(l)t}\right) + \mathcal{O}_p\left(\frac{N}{n^3}\right)\right) \\
& - \sum_{i=1}^{n} \left(E\left(X_{(k)i}X_{(l)i}\right) + \mathcal{O}_p\left(\frac{N}{n^2}\right)\right) \left(E\left(X_{(k')i}X_{(l')i}\right) + \mathcal{O}_p\left(\frac{N}{n^2}\right)\right)
\end{aligned}$$

$$\begin{aligned}
&\stackrel{\nabla}{=} \sum_{t=1}^{n} E\left(X_{(k)t}X_{(l)t}X_{(k')t}X_{(l')t}\right) - \sum_{i=1}^{n} E\left(X_{(k)i}X_{(l)i}\right) E\left(X_{(k')i}X_{(l')i}\right) + \mathcal{O}_p\left(\frac{N}{n^2}\right) \\
&= Cov\left(\sum_{t=1}^{n} X_{(k)t}X_{(l)t}, \sum_{t=1}^{n} X_{(k')t}X_{(l')t}\right) + \mathcal{O}_p\left(\frac{N}{n^2}\right).
\end{aligned}$$

To obtain the equality ∇, we use that $E\left(X_{(k)t}X_{(l)t}\right) = \mathcal{O}_p\left(\frac{1}{n}\right)$.
This implies immediately

$$Cov^*\left(\underline{T}_n^{LB}\right) = Cov\left(\underline{T}_n\right) + \mathcal{O}\left(\frac{N}{n}\right).$$

Asymptotic normality:

To prove the multivariate asymptotic normality of (4.4), it suffices to show for any real constants $\{c_{kl} : k, l \in \{1, \ldots, d\}\}$, it holds, as $n \to \infty$,

$$\underline{C}'\underline{T}_n^{LB} = \sqrt{n}\sum_{t=1}^{n} \underline{C}'\left(vec\left(\underline{X}_t^*\underline{X}_t^{*\prime}\right) - vec\left(\underline{X}_t\underline{X}_t'\right)\right) \xrightarrow{d} \mathcal{N}\left(0, \underline{C}'\underline{V}\ \underline{C}\right)$$

in probability, where $\underline{C} := vec\left(\left[c_{kl} : k, l \in \{1, \ldots, d\}\right]_{d\times d}\right)$. This is a consequence of the result of the Cramér-Wold Device (see e.g. Rao 1973).
It is shown that $\underline{V}$ is the asymptotic covariance matrix of $\underline{T}_n^{LB}$. Considering the linear combination $\underline{C}'\underline{T}_n^{LB}$, we have

$$Var^*\left(\underline{C}'\underline{T}_n^{LB}\right) \xrightarrow{p} \underline{C}'\underline{V}\ \underline{C}.$$

Lindeberg condition:

Now we consider the Lindeberg condition. Let $\zeta > 0$, $\delta = \Delta/8$ and $X_{(k)t}^* = \sum_{i=1}^{c} \frac{1}{\sqrt{n}}\theta_{ki,t}^*\varepsilon_{i,t}^*$, $k \in \{1, \ldots, d\}$ and $t \in \{1, \ldots, n\}$.

$$\begin{aligned}
Y_t^* : &= \sqrt{n}\underline{C}'\left(vec\left(\underline{X}_t^*\underline{X}_t^{*\prime}\right) - vec\left(\underline{X}_t\underline{X}_t{}'\right)\right) \\
&= \sqrt{n}\sum_{k,l=1}^{d} c_{kl}\left(X_{(k)t}^*X_{(l)t}^* - X_{(k)t}X_{(l)t}\right) \\
&= \frac{1}{\sqrt{n}}\sum_{k,l=1}^{d} c_{kl}\sum_{i,j=1}^{c}\left(\theta_{ki,t}^*\varepsilon_{i,t}^*\theta_{lj,t}^*\varepsilon_{j,t}^* - \theta_{ki,t}\varepsilon_{i,t}\theta_{lj,t}\varepsilon_{j,t}\right).
\end{aligned}$$

Then we compute

$$\begin{aligned}
&\sum_{t=1}^{n} E^*\left[Y_t^{*2}\mathbb{1}_{\{|Y_t^*|>\zeta\}}\right] \\
\leq &\sum_{t=1}^{n}\left(E^*\left[Y_t^{*2(1+\delta)}\right]\right)^{\frac{1}{1+\delta}}\left(E^*\left[\mathbb{1}_{\{|Y_t^*|>\zeta\}}\right]\right)^{\frac{\delta}{1+\delta}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n}\sum_{t=1}^{n}\left(E^*\left[\sum_{k,l=1}^{d} c_{kl}\sum_{i,j=1}^{c}\left(\theta^*_{ki,t}\varepsilon^*_{i,t}\theta^*_{lj,t}\varepsilon^*_{j,t}-\theta_{ki,t}\varepsilon_{i,t}\theta_{lj,t}\varepsilon_{j,t}\right)\right]^{2(1+\delta)}\right)^{\frac{1}{1+\delta}} \\
&\quad\cdot\left(E^*\left[\mathbb{1}_{\left\{\left|\sum_{k,l=1}^{d} c_{kl}\sum_{i,j=1}^{c}\left(\theta^*_{ki,t}\varepsilon^*_{i,t}\theta^*_{lj,t}\varepsilon^*_{j,t}-\theta_{ki,t}\varepsilon_{i,t}\theta_{lj,t}\varepsilon_{j,t}\right)\right|>\zeta\sqrt{n}\right\}}\right]\right)^{\frac{\delta}{1+\delta}} \\
&\le \frac{1}{n}\sum_{t=1}^{n}\left(E^*\left[\sum_{k,l=1}^{d} c_{kl}\sum_{i,j=1}^{c}\left(\theta^*_{ki,t}\varepsilon^*_{i,t}\theta^*_{lj,t}\varepsilon^*_{j,t}-\theta_{ki,t}\varepsilon_{i,t}\theta_{lj,t}\varepsilon_{j,t}\right)\right]^{2(1+\delta)}\right)^{\frac{1}{1+\delta}} \\
&\quad\cdot\left(E^*\left|\sum_{k,l=1}^{d} c_{kl}\sum_{i,j=1}^{c}\left(\theta^*_{ki,t}\varepsilon^*_{i,t}\theta^*_{lj,t}\varepsilon^*_{j,t}-\theta_{ki,t}\varepsilon_{i,t}\theta_{lj,t}\varepsilon_{j,t}\right)\right|^{2(1+\delta)}\right)^{\frac{\delta}{1+\delta}}\frac{1}{\zeta^{2\delta}n^{\delta}} \\
&\le \frac{1}{\zeta^{2\delta}n^{1+\delta}}\sum_{t=1}^{n}E^*\left|\sum_{k,l=1}^{d} c_{kl}\sum_{i,j=1}^{c}\left(\theta^*_{ki,t}\varepsilon^*_{i,t}\theta^*_{lj,t}\varepsilon^*_{j,t}-\theta_{ki,t}\varepsilon_{i,t}\theta_{lj,t}\varepsilon_{j,t}\right)\right|^{2(1+\delta)},
\end{aligned}
$$

where the first inequality is due to the Hölder inequality and the second one is due to the Markov inequality.

Analogously to the proof of Lemma 4.3, the following uniform consistency in k, l, i, j and t can be shown:

$$
E^*\left|\theta^*_{ki,t}\varepsilon^*_{i,t}\theta^*_{lj,t}\varepsilon^*_{j,t}\right|^{2(1+\delta)} = \theta_{ki,t}\theta_{lj,t}E\left|\varepsilon_{i,t}\varepsilon_{j,t}\right|^{2(1+\delta)} + o_p(1),
$$

in which $E\left|\varepsilon_{i,t}\right|^{8+\Delta} < \infty$ is required.

Together with $\theta_{ki,t} < \infty$, we have

$$
E^*\left[\sum_{k,l=1}^{d} c_{kl}\sum_{i,j=1}^{c}\left(\theta^*_{ki,t}\varepsilon^*_{i,t}\theta^*_{lj,t}\varepsilon^*_{j,t}-\theta_{ki,t}\varepsilon_{i,t}\theta_{lj,t}\varepsilon_{j,t}\right)\right]^{2(1+\delta)} < \infty,
$$

in probability, and therefore the Lindeberg condition is fulfilled. The Central Limit Theorem for independent triangular arrays yields the desired asymptotic normality.

□

4.3 Multivariate model with weakly dependent innovations

We consider a multivariate model with underlying varying volatility function and dependent innovations. In section 4.2.2, we have discussed a local resampling mechanism, which is able to deal with the underlying varying setting. A nature idea now is to combine the local bootstrap with other bootstrap procedures, which are able to capture the dependence structure. Following this idea, we discuss first the so-called local block bootstrap, introduced by Paparoditis and Politis (2002), and then propose a local dependent wild bootstrap procedure by applying the dependent wild bootstrap of Shao (2010) in nonoverlapping local windows.

4.3.1 Model and the assumptions

We consider the following discrete-time model for the multivariate intraday log-return process:

$$\underline{X}_t = \frac{1}{\sqrt{n}}\Theta\left(\frac{t}{n}\right)\underline{\xi}_t, \quad t = 1, \ldots, n. \tag{4.8}$$

Assumption.
(D1) *Θ denotes an $d \times c$ dimensional spot covolatility term. Its elements $\{\theta_{kl} : k = 1, \ldots, d; l = 1, \ldots, c\}$ can be described by non-stochastic continuous differentiable functions $\theta_{kl} : [0,1] \to (0, \infty)$ with a first derivative, which is bounded from above.*
(D2) *$\underline{\xi}_t$ denotes an c-dimensional vector of stationary time series with elements $\{(\xi_{i,t})_{t\in\mathbb{Z}} : i = 1, \ldots, c\}$, which are i.i.d. with $E\xi_{1,t} = 0$, $E\xi_{1,t}^2 = 1$ and $E|\xi_{1,t}|^{8+\Delta} < \infty$ for some $\Delta > 0$. Furthermore $\sum_{h=-\infty}^{\infty} |\gamma_{\xi_1}(h)| < \infty$ and $(\xi_{1,t})_{t\in\mathbb{Z}}$ satisfies GMC(2).*
(D3) *$\left(\xi_{1,t}^2\right)_{t\in\mathbb{Z}}$ is stationary with $\sum_{h=-\infty}^{\infty} \left|\gamma_{\xi_1^2}(h)\right| < \infty$*

Recall that $\Sigma(u) := \Theta(u)\Theta(u)'$, which is a $d \times d$ matrix. A central limit theorem based on this multivariate discrete-time model is given as follows:

Theorem 4.5. *For the discrete-time model* (4.8)*, it holds under assumptions (D1)-(D3), as $n \to \infty$, that*

$$\underline{\widetilde{T}}_n := \sqrt{n}\left(vec\left(\sum_{t=1}^{n} \underline{X}_t \underline{X}_t'\right) - vec\left(\int_0^1 \Sigma(u)du\right)\right) \xrightarrow{d} \mathcal{N}\left(0, \underline{\widetilde{V}}\right),$$

where $\underline{\widetilde{V}}$ *is an* $d^2 \times d^2$ *matrix with elements*

$$\begin{aligned}\widetilde{V}_{klk'l'} :&= \int_0^1 \sum_{i=1}^{c} \theta_{ki}(u)\theta_{li}(u)\theta_{k'i}(u)\theta_{l'i}(u)du \left(\sum_{h=-\infty}^{\infty} \gamma_{\xi_1^2}(h) - 2 \sum_{h=-\infty}^{\infty} \gamma^2_{\xi_1}(h) \right) \\ &+ \int_0^1 \left(\Sigma_{kk'}(u)\Sigma_{ll'}(u) + \Sigma_{kl'}(u)\Sigma_{k'l}(u)\right) du \sum_{h=-\infty}^{\infty} \gamma^2_{\xi_1}(h),\end{aligned}$$

$k, l, k', l' = 1, \ldots, d.$

Remark 4.6. *Similarly to the situation with independent innovations, we give a Central Limit Theorem for the component* $X_{(k),t}X_{(l),t}$ *of the vector* $vec\,(\underline{X}_t\underline{X}_t)$ *for the weakly dependent model.*
It holds under the same assumptions of Theorem 4.5, as $n \to \infty$, *that*

$$\widetilde{T}_{kl,n} := \sqrt{n} \left(\sum_{t=1}^{n} X_{(k),t}X_{(l),t} - \int_0^1 \Sigma_{kl}(u)du \right) \xrightarrow{d} \mathcal{N}\left(0, \widetilde{V}_{kl}\right),$$

where

$$\begin{aligned}\widetilde{V}_{kl} :&= \int_0^1 \sum_{i=1}^{c} \theta^2_{ki}(u)\theta^2_{li}(u) \left(\sum_{h=-\infty}^{\infty} \gamma_{\xi_1^2}(h) - 2 \sum_{h=-\infty}^{\infty} \gamma^2_{\xi_1}(h) \right) \\ &+ \int_0^1 \left(\Sigma_{kk}(u)\Sigma_{ll}(u) + \Sigma^2_{kl}(u)\right) du \sum_{h=-\infty}^{\infty} \gamma^2_{\xi_1}(h),\end{aligned}$$

$k, l = 1, \ldots, d.$
Ein consistent estimator for the asymptotic covariance is unknown. We cannot give a confidence interval via normal approximation.

We obtain that the asymptotic covariance $\underline{\widetilde{V}}$ depends on the the spot volatility Θ and the autocovariance of $(\xi_{1,t})_{t\in\mathbb{Z}}$ and $\left(\xi^2_{1,t}\right)_{t\in\mathbb{Z}}$. The proposed bootstrap methods need to be able to capture all these properties.

4.3.2 The local block bootstrap

The local block bootstrap introduced by Paparoditis and Politis (2002), and Dowla et al. (2003) is motivated from nonstationary process with slowly changing deterministic trend. See Dowal et al. (2013) for an application of this bootstrap procedure on a kernel regression for estimating deterministic trend functions. The bootstrap pseudo-observations are generated by drawing randomly with replacement from the blocks, which are close to the data point.

Bootstrap Procedure

Let realizations $\underline{X}_1, \ldots, \underline{X}_n$ be given. The local block bootstrap algorithm is precisely described by the following steps.

- **Step 1:** Add several data points at the left side and the right side of the original date set as Step 1 by the local bootstrap.
- **Step 2:** Select a number $B \in \mathbb{N}$, which denotes the block size, and select a number $N \in \mathbb{N}$. $2N+1$ is the size of the local window. Let $b = \left\lceil \frac{n}{B} \right\rceil$, which is the number of blocks.
- **Step 3:** For $s = 0, 1, \ldots, b-1$ and $i = 1, \ldots, B$, define the bootstrap pseudo series $\underline{X}_1^*, \ldots, \underline{X}_{bB}^*$ by $\underline{X}_{sB+i}^* := \underline{X}_{I_s^*+sB+i}$, in which I_s^* ∼Laplace on the set $\{-N, -N+1, \ldots, N\}$.
- **Step 4:** Based on the bootstrap sample $\underline{X}_1^*, \ldots, \underline{X}_n^*$, the bootstrap realized covariance is defined as

$$RCV_n^{LBB} := \sum_{t=1}^{n} \underline{X}_t^* \underline{X}_t^{*\prime}.$$

Step 1 help us to deal with the boundary problem, if for some s and i, $I_s^* + sB + i$ is outside the range of integers 1 to n. In this case, the dependence structure of the data in the first $\left\lceil \frac{N}{B} \right\rceil$ blocks and in the last $b - \left\lceil \frac{n-N}{B} \right\rceil$ blocks may not be the same as the structure of the original data. But the probability, that a data point is selected, is for each data point the same and equals $\frac{1}{2N+1}$. The bootstrap expectation value $E^* RCV_n^{LBB}$ can be explicitly given as a $d \times d$ matrix as follows:

$$\begin{aligned} E^* RCV_n^{LBB} &= \left\{ \sum_{t=1}^{n} E^* X_{(k)t}^* X_{(l)t}^* \right\}_{l,k=1,\ldots,d} = \left\{ \sum_{t=1}^{n} \sum_{j=t-N}^{t+N} \frac{1}{2N+1} X_{(k)j} X_{(l)j} \right\}_{l,k=1,\ldots,d} \\ &= \left\{ \sum_{t=1}^{n} X_{(k)t} X_{(l)t} \right\}_{l,k=1,\ldots,d} = \sum_{t=1}^{n} \underline{X}_t \underline{X}_t{}'. \end{aligned} \tag{4.9}$$

The values B and $2N+1$ correspond to the bootstrap block size and the local window size respectively and all depend on n. We need to choose appropriate rates of them to ensure that the stochastic structure of the log-returns is correctly imitated. The local window size need to be small enough to neglect the varying volatility structure within a window, but large enough for applying the block bootstrap method. We state them as the following assumption:

Assumption.
(D4) *Let* $B = \mathcal{O}\left(n^{\eta_1}\right)$ *and* $N = \mathcal{O}\left(n^{\eta_2}\right)$, *where* $\eta_1, \eta_2 \in (0,1)$. *We assume* $\eta_1 + \eta_2 < 1$ *and* $\eta_1 < \frac{\eta_2}{2}$.
This ensures that $\frac{BN}{n} \to 0$ *and* $\frac{B}{\sqrt{N}} \to 0$, *which lead to a neglect of the bias of the bootstrap asymptotic covariance. A sequence of numbers for* η_1 *and* η_2 *which satisfy this assumption can be easy given.*

Validity of the Bootstrap
The following Lemma give us the convergence rate of the bootstrap covariance:

Lemma 4.7. *Let* $\{\underline{X}_t : t = 1, \ldots, n\}$ *be given by the discrete-time model 4.8 and the assumptions (D1)-(D4) be fulfilled. Let* $\{\underline{X}_t^* : t = 1, \ldots, n\}$ *be estimated via the local block bootstrap as described above. It holds uniformly in* i, j, *when* $\left\lceil \frac{i}{B} \right\rceil - \left\lceil \frac{j}{B} \right\rceil = 0$, *and for any fixed* $k, l, k', l' \in \{1, \ldots, n\}$, *that*

$$Cov^*\left(X^*_{(k)i} X^*_{(l)i}, X^*_{(k')j} X^*_{(l')j}\right) = Cov\left(X_{(k)i} X_{(l)i}, X_{(k')j} X_{(l')j}\right) + \mathcal{O}_p\left(n^{\eta_2 - 3}\right) + \mathcal{O}_p\left(n^{-2-\eta_2/2}\right).$$

This result shows that the bootstrap pseudo observations have the same asymptotic covariance as the original observations. Based on this result, a Central Limit Theorem for the bootstrap time series is given as follows:

Theorem 4.8. *Let* $\{\underline{X}_t : t = 1, \ldots, n\}$ *be given by the discrete-time model* (4.8) *and the assumptions D1-D4 be fulfilled. Let* $\{\underline{X}_t^* : t = 1, \ldots, n\}$ *be estimated via the local block bootstrap as described above. It holds true, as* $n \to \infty$, *that*

$$\underline{T}_n^{LBB} := \sqrt{n}\left(vec\left(RCV_n^{LBB}\right) - vec\left(E^* RCV_n^{LBB}\right)\right) \xrightarrow{d} \mathcal{N}\left(0, \underline{\widetilde{V}}\right),$$

in probability, where $E^* RCV_n^{LBB}$ *is given by* (4.9) *and* $\underline{\widetilde{V}}$ *is given in Theorem 4.5. This result implies the validity of the local block bootstrap procedure:*

$$\sup_{x \in \mathbb{R}^{d^2}} \left| P(\underline{T}_n^{LBB} \le x) - P(\underline{\widetilde{T}}_n \le x) \right| \xrightarrow{p} 0.$$

4.3.3 The local dependent wild bootstrap

The dependent wild bootstrap introduced by Shao (2010) is an extension of the traditional wild bootstrap of Wu (1986) to the time series with dependent setting. The bootstrap pseudo-observations are generated by multiplying each centered original observation of the time series by an external random variable, which comes from

a stationary process with zero mean, unit variance and covariance, which is a kernel function. The consistency of this bootstrap method is established for regularly spaced time series as well as for irregularly time series by Shao (2010). Comparison to some block bootstrap procedures can be also found there.
We propose a local dependent wild bootstrap procedure by applying the dependent wild bootstrap in nonoverlapping local windows for our log-return model with weak dependent innovations and varying volatility structure.

Bootstrap Procedure
Given realizations $\underline{X}_1, \ldots, \underline{X}_n$. Let $\underline{Y}_i = \underline{X}_i \underline{X}_i'$, $i = 1, \ldots, n$. The local dependent wild bootstrap algorithm is described by the following steps.

- **Step 1:** Select a number $H \in \mathbb{N}$, which denotes the size of a local window. For simplicity of notation we want $m = \dfrac{n}{H}$, which denotes the number of the local windows, to be positive integer.

- **Step 2:** Generate independently for each $s \in \{0, \ldots, m-1\}$ a group of pseudo-random numbers $W_{sH+1}, W_{sH+2}, \ldots, W_{(s+1)H}$ satisfying the assumption *(D6)* (see later). The bootstrap observations $\underline{Y}_1^*, \ldots, \underline{Y}_n^*$ are defined as

$$\underline{Y}_{sH+i}^* := \overline{\underline{Y}}_s + \left(\underline{Y}_{sH+i} - \overline{\underline{Y}}_s\right) W_{sH+i},$$

  $s = 0, \ldots, m-1$, $i = 1, \ldots, H$, where $\overline{\underline{Y}}_s = \dfrac{1}{H} \sum_{j=1}^{H} \underline{Y}_{sH+j}$.

- **Step 3:** Based on the bootstrap observations $\underline{Y}_1^*, \ldots, \underline{Y}_n^*$, the bootstrap realized covariance is defined as

$$RCV_n^{LDW} := \sum_{t=1}^{n} \underline{Y}_t^* = \sum_{s=0}^{m-1} \sum_{i=1}^{H} \left(\overline{\underline{Y}}_s + \left(\underline{Y}_{sH+i} - \overline{\underline{Y}}_s\right) W_{sH+i}\right).$$

The bootstrap expectation value $E^* RCV_n^{LDW}$ is given as follows:

$$\begin{aligned} E^* RCV_n^{LDW} &= E^* \sum_{t=1}^{n} \underline{Y}_t^* = \sum_{s=0}^{m-1} \sum_{i=1}^{H} \left(\overline{\underline{Y}}_s + \left(\underline{Y}_{sH+i} - \overline{\underline{Y}}_s\right) E W_{sH+i}\right) \\ &= \sum_{s=0}^{b-1} \sum_{i=1}^{H} \overline{\underline{Y}}_s = \sum_{t=1}^{n} \underline{Y}_t = \sum_{t=1}^{n} \underline{X}_t \underline{X}_t'. \end{aligned} \tag{4.10}$$

The dependent wild bootstrap is motivated from the lag-window spectrum estimators. Under some assumptions on W, the bootstrap covariance in the sample mean case is equivalent to the lag window estimator at zero frequency, with which the sum of the autocovariance can be consistently estimated. The assumptions on W were given by Shao under the framework of a stationary time series. These assumptions are adapted for our nonstationary case. An assumption on the local window size H is also stated in the following:

Assumption.
(D5) *Let* $H = \mathcal{O}(n^{\eta_3})$*, where* $\eta_3 \in (0, 1/2)$*. It ensures that* $\frac{H^2}{n} = o(1)$*.*
(D6) *Let* $(W_t)_{t\in\mathbb{Z}}$ *be a stationary real-valued time series with* $EW_t = 0$*,* $EW_t^2 = 1$*,* $Cov(W_i, W_j) = K\left(\frac{i-j}{l}\right)$*, where* $K(\cdot)$ *is a kernel function, and* $EW_t^{2+\delta} < \infty$ *for some* $\delta > 0$*. Furthermore* $\int_{-\infty}^{\infty} K(u)e^{-iux}du \geq 0$*, for* $x \in \mathbb{R}$*, which ensures the nonnegative definiteness of the covariance matrix of* $W_1, \ldots, W_n$*.*
$l = \mathcal{O}(n^{\eta_4})$ *is a bandwidth parameter. Assume that* $\eta_4 < \frac{\eta_3}{3}$*, which ensures that* $\frac{1}{l} + \frac{l}{H^{1/3}} = o(1)$*.*
(D7) *Assume that* $\sum_{h=1}^{\infty} h^2 |\gamma_{\xi_1^2}(h)| < \infty$*.*

A few commonly used kernel functions, e.g. Parzen window, satisfy assumption (*D6*). The bandwidth parameter l plays a similar role as the block size in the block bootstrap methods. Together with assumption *(D5)* and *(D6)*, we have $\frac{Hl}{n} = o(1)$, which is similar to $\frac{BN}{n} = o(1)$ in the local block bootstrap.

Validity of the Bootstrap

A Central Limit Theorem for the bootstrap time series based on the local dependent wild bootstrap is given in the following, and this implies the validity of the bootstrap procedure.

Theorem 4.9. *Let* $\{\underline{X}_t : t = 1, \ldots, n\}$ *be given by the discrete-time model* (4.8) *and the assumptions (D1)-(D3) and (D5)-(D7) be fulfilled. let* RCV_n^* *be estimated via the local dependent wild bootstrap as described above. It holds true, as* $n \to \infty$*, that*

$$\underline{T}_n^{LDW} := \sqrt{n}\left(vec\left(RCV_n^{LDW}\right) - vec\left(E^*RCV_n^{LDW}\right)\right) \xrightarrow{d} \mathcal{N}\left(0, \widetilde{\underline{V}}\right),$$

in probability, where $E^*RCV_n^{LDW}$ *is given by* (4.10) *and* $\widetilde{\underline{V}}$ *is given in Theorem 4.5. The result implies the validity of the local dependent wild bootstrap procedure:*

$$\sup_{x\in\mathbb{R}^{d^2}} |P(\underline{T}_n^{LDW} \leq x) - P(\widetilde{\underline{T}}_n \leq x)| \xrightarrow{p} 0.$$

4.3.4 A simulation study

We consider again a bivariate model with the spot volatility function Θ, which is given in section 4.2.3, and we choose the same innovation term as in section 3.3, that means $\{\xi_{i,t} : t = 1, \dots, n\}$ are generated by the $MA(1)$ process

$$\xi_{i,t} = a_1 e_{i,t-1} + e_{i,t}, \quad i = 1, 2,$$

where $e_{i,t}$ is i.i.d. sequence of $\mathcal{N}(0, 1/(1 + a_1^2))$, for $n = 200$ and $a_1 = 0.5$. The statistics of interest is $T_{12,n}$ (see Remark 4.6 for the asymptotic distribution).

With this simple model, we want to give a first impression of the effectiveness of both bootstrap procedures in finite sample ($n = 200$) case. For the local block bootstrap (LB bootstrap), we choose the local windows size $2N + 1 = 31$ and the block size $B = 5$, and for the local dependent wild bootstrap (LDW bootstrap), we choose the local windows size $H = 10$ and the bandwidth parameter $l = 2$. In a small sample case, there are not many choices left by choosing the parameter values, especially for the bandwidth l, because of the local resampling idea. The sensitivity to the choice of the block length and the bandwidth needs to be discussed in a large sample situation. The bootstrap procedures and the whole simulation are both repeated 500 times to obtain boxplots of bootstrapped quantiles, which are given in Figure 4.5.

The boxplots on the left side of each panel give the bootstrap quantiles via LB bootstrap, while the right ones give the results obtained from the LDW bootstrap. It can be easily seen that the medians of LB bootstrap boxplots are closer to the true quantiles. The LDW bootstrap quantiles vary in a smaller range, but the medians of its boxplots stay far away from the true quantiles. In this study, both bootstrap methods overestimate the true covariance.

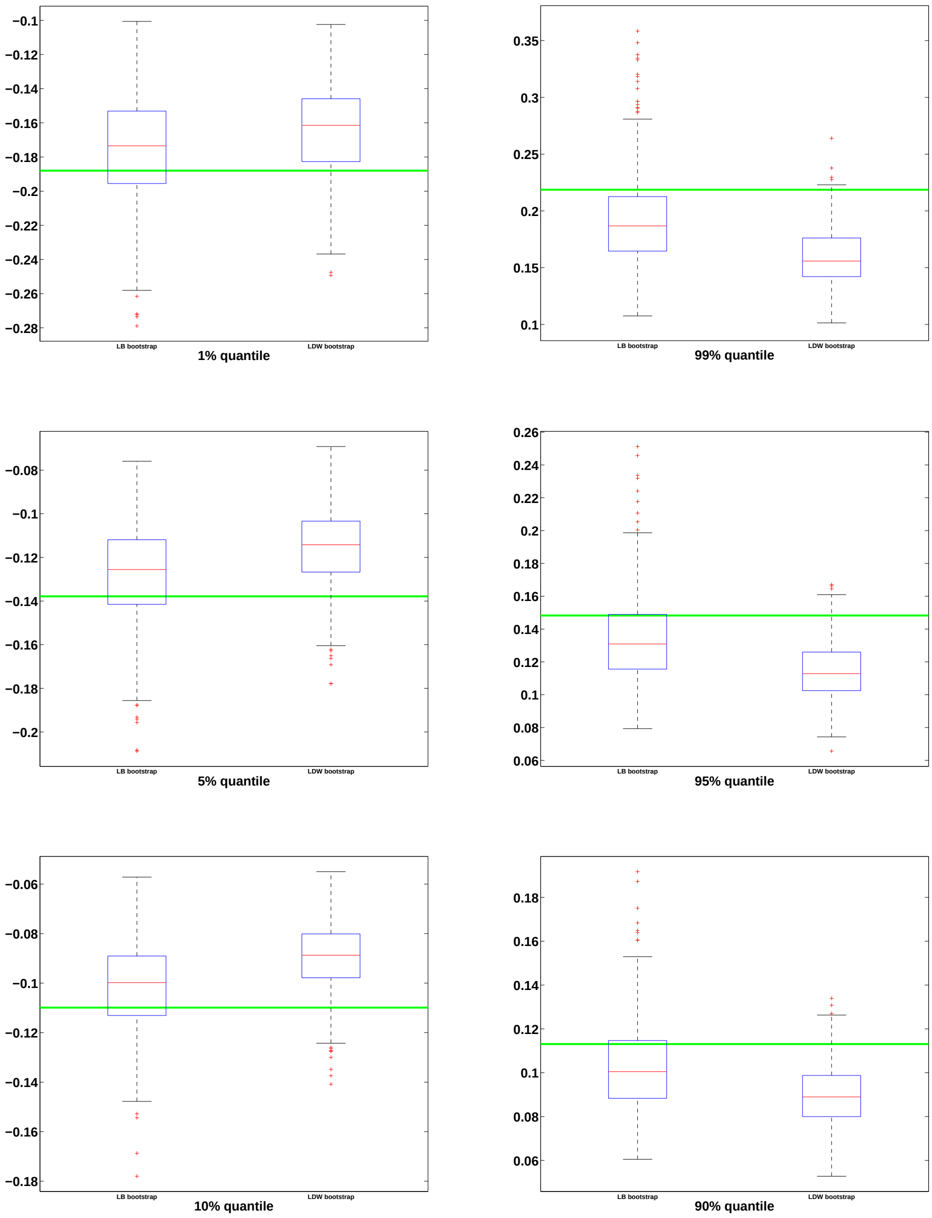

Figure 4.5: Boxplots of bootstraped quantiles

4.3.5 Proofs and auxiliary results

To simplify the notation, we use $\theta_{ki,t}$ instead of $\theta_{ki}\left(\frac{t}{n}\right)$ and $\Sigma_{ki,t}$ instead of $\Sigma_{ki}\left(\frac{t}{n}\right)$ in the following proofs.

Lemma 4.10 below is required for the proof of Theorem 4.5.

Lemma 4.10. *Let $\{Y_t : t \in \mathbb{Z}\}$ be a stationary process with autocovariances $\{\gamma_Y(h) : h \in \mathbb{Z}\}$ and $\sum_{h=-\infty}^{\infty} \gamma_Y(h) < \infty$. Let assumption (D1) be fulfilled. It holds, as $n \to \infty$, that*

$$\frac{1}{n} \sum_{t,t'=1}^{n} \sum_{i,j,p,q=1}^{c} \theta_{ki,t}\theta_{lj,t}\theta_{k'p,t'}\theta_{l'q,t'}\gamma_Y(t-t')$$
$$\longrightarrow \int_0^1 \sum_{i,j,p,q=1}^{c} \theta_{ki}(u)\theta_{lj}(u)\theta_{k'p}(u)\theta_{l'q}(u)du \sum_{h=-\infty}^{\infty} \gamma_Y(h).$$

Proof. Under assumption *(D1)* and due to the mean value theorem, there exists $c1_t, c2_t \in \left(\frac{t'}{n}, \frac{t}{n}\right)$ (say $t > t'$), that it holds uniformly in $t, t' \in \{1, \ldots, n\}$

$$\begin{aligned}
\theta_{ki,t}\theta_{lj,t}\theta_{k'p,t'}\theta_{l'q,t'} &= \theta_{ki,t}\theta_{lj,t}\left(\theta_{k'p,t} + \frac{t-t'}{n}\theta'_{k'p}(c1_t)\right)\left(\theta_{l'q,t} + \frac{t-t'}{n}\theta'_{l'q}(c2_t)\right) \\
&= \theta_{ki,t}\theta_{lj,t}\theta_{k'p,t}\theta_{l'q,t} + \frac{t-t'}{n}\theta_{ki,t}\theta_{lj,t}\theta_{l'q,t}\theta'_{k'p}(c1_t) \\
&\quad + \frac{t-t'}{n}\theta_{ki,t}\theta_{lj,t}\theta_{k'p,t}\theta'_{l'q}(c2_t) + \left(\frac{t-t'}{n}\right)^2 \theta_{ki,t}\theta_{lj,t}\theta'_{k'p}(c1_t)\theta'_{l'q}(c2_t).
\end{aligned}$$

where $\theta'(\cdot)$ denote the We consider the first term and obtain

$$\begin{aligned}
&\frac{1}{n} \sum_{t,t'=1}^{n} \sum_{i,j,p,q=1}^{c} \theta_{ki,t}\theta_{lj,t}\theta_{k'p,t}\theta_{l'q,t}\gamma_Y(t-t') \\
= &\frac{1}{n} \sum_{h=-(n-1)}^{n-1} \sum_{t=1}^{n-|h|} \sum_{i,j,p,q=1}^{c} \theta_{ki,t}\theta_{lj,t}\theta_{k'p,t}\theta_{l'q,t}\gamma_Y(h) \\
= &\underbrace{\frac{1}{n} \sum_{h=-(n-1)}^{n-1} \gamma_Y(h) \sum_{t=1}^{n} \sum_{i,j,p,q=1}^{c} \theta_{ki,t}\theta_{lj,t}\theta_{k'p,t}\theta_{l'q,t}}_{A} \\
&- \underbrace{\frac{1}{n} \sum_{h=-(n-1)}^{n-1} \gamma_Y(h) \sum_{t=n-|h|+1}^{n} \sum_{i,j,p,q=1}^{c} \theta_{ki,t}\theta_{lj,t}\theta_{k'p,t}\theta_{l'q,t},}_{B}
\end{aligned}$$

where as $n \to \infty$,

$$A \longrightarrow \int_0^1 \sum_{i,j,p,q=1}^{c} \theta_{ki}(u)\theta_{lj}(u)\theta_{k'p}(u)\theta_{l'q}(u)du \sum_{h=-\infty}^{\infty} \gamma_Y(h).$$

Further, since $\sum_{h=-\infty}^{\infty} \gamma_Y(h) < \infty$ and $\theta_{kj,t} < \infty$ for all k, j, t, due to the Kronecker-Lemma,

$$B \leq \frac{1}{n} \sum_{h=-(n-1)}^{n-1} \gamma_Y(h)|h| \sup_{t \in \{1,\ldots,n\}} \sum_{i,j,p,q=1}^{c} \theta_{ki,t}\theta_{lj,t}\theta_{k'p,t}\theta_{l'q,t} \longrightarrow 0,$$

This leads to

$$\frac{1}{n}\sum_{t,t'=1}^{n}\sum_{i,j,p,q=1}^{c}\theta_{ki,t}\theta_{lj,t}\theta_{k'p,t}\theta_{l'q,t}\gamma_Y(t-t')$$
$$\longrightarrow \int_0^1 \sum_{i,j,p,q=1}^{c}\theta_{ki}(u)\theta_{lj}(u)\theta_{k'p}(u)\theta_{l'q}(u)\sum_{h=-\infty}^{\infty}\gamma_Y(h).$$

We consider a further term. Again, due to the Kronecker- Lemma, as $n\to\infty$,

$$\begin{aligned}
&\frac{1}{n}\sum_{t,t'=1}^{n}\sum_{i,j,p,q=1}^{c}\frac{t-t'}{n}\theta_{ki,t}\theta_{lj,t}\theta_{l'q,t}\theta'_{k'p}(c1_t)\gamma_Y(t-t')\\
=\;&\frac{1}{n}\sum_{h=-(n-1)}^{n-1}\sum_{t=1}^{n-|h|}\sum_{i,j,p,q=1}^{c}\frac{h}{n}\theta_{ki,t}\theta_{lj,t}\theta_{l'q,t}\theta'_{k'p}(c1_t)\gamma_Y(h)\\
\leq\;&\frac{1}{n}\sum_{h=-(n-1)}^{n-1}\gamma_Y(h)|h|\sup_{t\in\{1,\ldots,n\}}\sum_{i,j,p,q=1}^{c}\theta_{ki,t}\theta_{lj,t}\theta_{l'q,t}\theta'_{k'p}(c1_t)\longrightarrow 0.
\end{aligned}$$

Analogously, it can be shown as $n\to\infty$, that

$$\frac{1}{n}\sum_{t,t'=1}^{n}\sum_{i,j,p,q=1}^{c}\frac{t-t'}{n}\theta_{ki,t}\theta_{lj,t}\theta_{k'p,t}\theta'_{l'q}(c2_t)\gamma_Y(t-t')\longrightarrow 0,$$

and

$$\frac{1}{n}\sum_{t,t'=1}^{n}\sum_{i,j,p,q=1}^{c}\left(\frac{t-t'}{n}\right)^2\theta_{ki,t}\theta_{lj,t}\theta'_{k'p,t}(c1_t)\theta'_{l'q}(c2_t)\gamma_Y(t-t')\longrightarrow 0,$$

which conclude the proof. □

Proof of Theorem 4.5

We consider the asymptotic behavior of

$$\underline{T_n} := \sqrt{n}\left(\sum_{t=1}^{n} vec\left(\underline{X}_t\underline{X}'_t\right)-\sum_{t=1}^{n} vec\left(E\underline{X}_t\underline{X}'_t\right)\right) \tag{4.11}$$

in two steps. At first the asymptotic covariance matrix of $\underline{T_n}$ will be estimated and then a multivariate asymptotic normality will be proved. The assertion follows with the Lemma of Slutsky.

Asymptotic covariance:

Recalling the definition of the log-price process and the assumption *(D2)* that $\{(\xi_{i,t})_{t\in\mathbb{Z}} : i=1,\ldots,c\}$ are i.i.d. stationary time series with $E\xi_{1,t}=0$ and $E\xi_{1,t}^2=1$, we have that

$$E\left(X_{(k)t}X_{(l)t}\right) \;=\; \frac{1}{n}E\left(\sum_{i=1}^{c}\theta_{ki,t}\xi_{i,t}\sum_{j=1}^{c}\theta_{lj,t}\xi_{j,t}\right)$$

$$= \frac{1}{n}\sum_{i,j=1}^{c}\theta_{ki,t}\theta_{lj,t}E\left(\xi_{i,t}\xi_{j,t}\right) = \frac{1}{n}\sum_{i=1}^{c}\theta_{ki,t}\theta_{li,t} = \frac{1}{n}\Sigma_{kl,t} \qquad (4.12)$$

and therefore $vec\left(E\underline{X}_t\underline{X}_t'\right) = \frac{1}{n}vec\left(\Sigma\left(\frac{t}{n}\right)\right)$.

The elements of the covariance matrix is computed in the following:

$$\begin{aligned}
V'_{klk'l',n} &:= Cov\left(\sqrt{n}\sum_{t=1}^{n}X_{(k)t}X_{(l)t}, \sqrt{n}\sum_{t'=1}^{n}X_{(k')t'}X_{(l')t'}\right)\\
&= n\sum_{t,t'=1}^{n}Cov\left(X_{(k)t}X_{(l)t}, X_{(k')t'}X_{(l')t'}\right)\\
&= n\sum_{t,t'=1}^{n}\left(E\left(X_{(k)t}X_{(l)t}X_{(k')t'}X_{(l')t'}\right) - E\left(X_{(k)t}X_{(l)t}\right)E\left(X_{(k')t'}X_{(l')t'}\right)\right)\\
&= \frac{1}{n}\sum_{t,t'=1}^{n}\sum_{i,j,p,q=1}^{c}\theta_{ki,t}\theta_{lj,t}\theta_{k'p,t'}\theta_{l'q,t'}E\left(\xi_{i,t}\xi_{j,t}\xi_{p,t'}\xi_{q,t'}\right) - \frac{1}{n}\sum_{t,t'=1}^{n}\Sigma_{kl,t}\Sigma_{k'l',t'}\\
&= \frac{1}{n}\sum_{t,t'=1}^{n}A_{tt'} - \frac{1}{n}\sum_{t,t'=1}^{n}\Sigma_{kl,t}\Sigma_{k'l',t'},
\end{aligned}$$

where $A_{tt'}$ is the obvious notation. Due to the assumption *(D2)*, we have

$$\begin{aligned}
A_{tt'} &= \left(\sum_{i=1}^{c}\theta_{ki,t}\theta_{li,t}\theta_{k'i,t'}\theta_{l'i,t'}E\xi_{i,t}^2\xi_{i,t'}^2\right) + \left(\sum_{i,p=1,i\neq p}^{c}\theta_{ki,t}\theta_{li,t}\theta_{k'p,t'}\theta_{l'p,t'}E\xi_{i,t}^2E\xi_{p,t'}^2\right)\\
&\quad + \left(\sum_{i,j=1,i\neq j}^{c}\theta_{ki,t}\theta_{lj,t}\theta_{k'i,t'}\theta_{l'j,t'}E\xi_{i,t}\xi_{i,t'}E\xi_{j,t}\xi_{j,t'}\right)\\
&\quad + \left(\sum_{i,j=1,i\neq j}^{c}\theta_{ki,t}\theta_{lj,t}\theta_{k'j,t'}\theta_{l'i,t'}E\xi_{i,t}\xi_{i,t'}E\xi_{1,t}\xi_{1,t'}\right)\\
&= \left(\sum_{i=1}^{c}\theta_{ki,t}\theta_{li,t}\theta_{k'i,t'}\theta_{l'i,t'}E\xi_{i,t}^2\xi_{i,t'}^2\right) + \left(\Sigma_{kl,t}\Sigma_{k'l',t'} - \sum_{i=1}^{c}\theta_{ki,t}\theta_{li,t}\theta_{k'i,t'}\theta_{l'i,t'}\right)\\
&\quad + \left(\left(\Sigma_{kk',t}\Sigma_{ll',t'} - \sum_{i=1}^{c}\theta_{ki,t}\theta_{li,t}\theta_{k'i,t'}\theta_{l'i,t'}\right)\left(E\xi_{1,t}\xi_{1,t'}\right)^2\right)\\
&\quad + \left(\left(\Sigma_{kl',t}\Sigma_{k'l,t'} - \sum_{i=1}^{c}\theta_{ki,t}\theta_{li,t}\theta_{k'i,t'}\theta_{l'i,t'}\right)\left(E\xi_{1,t}\xi_{1,t'}\right)^2\right)\\
&= \sum_{i=1}^{c}\theta_{ki,t}\theta_{li,t}\theta_{k'i,t'}\theta_{l'i,t'}\left(\gamma_{\xi_1^2}\left(t-t'\right) - 2\gamma_{\xi_1}^2\left(t-t'\right)\right) + \Sigma_{kl,t}\Sigma_{k'l',t'}\\
&\quad + \left(\Sigma_{kk',t}\Sigma_{l'l,t'} + \Sigma_{kl',t}\Sigma_{k'l,t'}\right)\gamma_{\xi_1}^2\left(t-t'\right),
\end{aligned}$$

and then obtain

$$V'_{klk'l',n} = \frac{1}{n}\sum_{t,t'=1}^{n}A_{tt'} - \frac{1}{n}\sum_{t,t'=1}^{n}\Sigma_{kl,t}\Sigma_{k'l',t'}$$

$$
\begin{aligned}
= \quad & \frac{1}{n} \sum_{t,t'=1}^{n} \left[\sum_{i=1}^{c} \theta_{ki,t} \theta_{li,t} \theta_{k'i,t'} \theta_{l'i,t'} \left(\gamma_{\xi_1^2} \left(t - t'\right) - 2\gamma_{\xi_1}^2 \left(t - t'\right) \right) \right. \\
& \left. + \left(\Sigma_{kk',t} \Sigma_{ll',t'} + \Sigma_{kl',t} \Sigma_{k'l,t'} \right) \gamma_{\xi_1}^2 \left(t - t'\right) \right]
\end{aligned}
$$

According to Lemma 4.10 and since the covariances of the process ξ and the covariances of the process ξ^2 are summable, we obtain that, as $n \to \infty$

$$
V'_{klk'l',n} \longrightarrow \widetilde{V}_{klk'l'}.
$$

Asymptotic normality:
To prove the multivariate asymptotic normality of (4.11), we need to show that for any real constants $\{c_{kl} : k, l \in \{1, \ldots, d\}\}$, it holds, as $n \to \infty$, that

$$
\underline{C}' \underline{\widetilde{T}}_n = \sqrt{n} \sum_{t=1}^{n} \underline{C}' \left(vec\left(\underline{X}_t \underline{X}_t' \right) - vec\left(\frac{1}{n} \Sigma\left(\frac{t}{n} \right) \right) \right) \xrightarrow{d} \mathcal{N}\left(0, \underline{C}' \underline{\widetilde{V}} \underline{C} \right), \tag{4.13}
$$

where $\underline{C} := vec\left(\left[c_{kl} : k, l \in \{1, \ldots, d\} \right]_{d \times d} \right)$.

It is shown, that $\underline{\widetilde{V}}$ is the asymptotic covariance matrix of $\underline{\widetilde{T}}_n$. Considering the linear combination $\underline{C}' \underline{\widetilde{T}}_n$, we have

$$
Var\left(\underline{C}' \underline{\widetilde{T}}_n \right) \longrightarrow \underline{C}' \underline{\widetilde{V}} \underline{C}.
$$

Let

$$
\begin{aligned}
Y_t : \quad &= \quad \sqrt{n} \underline{C}' \left(vec\left(\underline{X}_t \underline{X}_t' \right) - vec\left(\frac{1}{n} \Sigma\left(\frac{t}{n} \right) \right) \right) \\
&= \quad \frac{1}{\sqrt{n}} \sum_{k,l=1}^{d} c_{kl} \left(\sum_{i,j=1}^{c} \theta_{ki,t} \xi_{i,t} \theta_{lj,t} \xi_{j,t} - \Sigma_{kl,t} \right).
\end{aligned}
$$

$(Y_t)_{t=1,\ldots,n}$ is a triangular scheme of random variables with $EY_t = 0$ and $\sum_{t=1}^{n} EY_t^2 < \infty$, which is shown above by the convergence of the variance. It is worth to mention that $(Y_t)_{t=1,\ldots,n}$ is not stationary.

Lindeberg condition: Let $\zeta > 0$. We have

$$
\begin{aligned}
\sum_{t=1}^{n} E\left[Y_t^2 \mathbb{1}_{\{|Y_t| > \zeta\}} \right] \quad &\leq \quad \sum_{t=1}^{n} \left(E\left[Y_t^4 \right] \right)^{\frac{1}{2}} \left(E\left[\mathbb{1}_{\{|Y_t| > \zeta\}} \right] \right)^{\frac{1}{2}} \\
&= \quad \frac{1}{n} \sum_{t=1}^{n} \left(E\left[\sum_{k,l=1}^{d} c_{kl} \left(\sum_{i,j=1}^{c} \theta_{ki,t} \xi_{i,t} \theta_{lj,t} \xi_{j,t} - \Sigma_{kl,t} \right) \right]^4 \right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
& \cdot\left(E\left[\mathbb{1}_{\left\{\left|\sum_{k,l=1}^{d} c_{kl}\left(\sum_{i,j=1}^{c} \theta_{ki,t}\xi_{i,t}\theta_{lj,t}\xi_{j,t}-\Sigma_{kl,t}\right)\right|>\zeta\sqrt{n}\right\}}\right]\right)^{\frac{1}{2}} \\
\leq\ & \frac{1}{n}\sum_{t=1}^{n}\left(E\left[\sum_{k,l=1}^{d} c_{kl}\left(\sum_{i,j=1}^{c} \theta_{ki,t}\xi_{i,t}\theta_{lj,t}\xi_{j,t}-\Sigma_{kl,t}\right)\right]^{4}\right)^{\frac{1}{2}} \\
& \cdot\left(E\left|\sum_{k,l=1}^{d} c_{kl}\left(\sum_{i,j=1}^{c} \theta_{ki,t}\xi_{i,t}\theta_{lj,t}\xi_{j,t}-\Sigma_{kl,t}\right)\right|^{4}\right)^{\frac{1}{2}} \frac{1}{\zeta^{2}n} \\
=\ & \frac{1}{\zeta^{2}n^{2}}\sum_{t=1}^{n} E\left[\sum_{k,l=1}^{d} c_{kl}\left(\sum_{i,j=1}^{c} \theta_{ki,t}\xi_{i,t}\theta_{lj,t}\xi_{j,t}-\Sigma_{kl,t}\right)\right]^{4},
\end{aligned}
$$

where the first inequality is due to the Cauchy-Schwarz inequality and the second one is due to the Markov inequality.
Since that the 8th moment of $\xi_{i,t}$ exists for all $i \in \{1,\dots,c\}$ and $t \in \mathbb{Z}$, we have

$$
E\left[\sum_{k,l=1}^{d} c_{kl}\left(\sum_{i,j=1}^{c} \theta_{ki,t}\xi_{i,t}\theta_{lj,t}\xi_{j,t}-\Sigma_{kl,t}\right)\right]^{4} < \infty
$$

and therefore as $n \to \infty$, that

$$
\sum_{t=1}^{n} E\left[Y_t^2 \mathbb{1}_{\{|Y_t|>\zeta\}}\right] \longrightarrow 0.
$$

Weak dependance conditions:
Recall $\xi_{i,t} = H(\dots,\varepsilon_{i,t-1},\varepsilon_{i,t})$ for all $i \in \{1,\dots,c\}$ and $t \in \mathbb{Z}$. Let $\left(\varepsilon'_{i,t}\right)_{t\in\mathbb{Z}}$ be an i.i.d. copy of $(\varepsilon_{i,t})_{t\in\mathbb{Z}}$. Let $\widetilde{\xi}_{i,t}^{(r)}$ be a coupled version of $\xi_{i,t}$ with $\varepsilon_{i,s}$ being replaced by $\varepsilon'_{i,s}$ for all $s \leq t-r$. The $GMC(2)$ property of $(\xi_{i,t})_{t\in\mathbb{Z}}$ leads to that there exist $C > 0$ and $0 < \vartheta < 1$ such that for all $t \in \mathbb{N}$, $\sqrt{E\left(\xi_{i,t}-\widetilde{\xi}_{i,t}^{(r)}\right)^2} \leq C\vartheta^r$.

For all $u \in \mathbb{N}$, all indices $1 \leq s_1 < s_2 < \dots < s_u < s_u + r = t_1 \leq t_2 \leq n$ and for all measurable square-integrable functions $g : \mathbb{R}^u \to \mathbb{R}$ with $\|g\|_\infty = \sup_{x\in\mathbb{R}^u} |g(x)| \leq 1$, we have

$$
\begin{aligned}
& \left|Cov\left(g\left(Y_{s_1},\dots,Y_{s_u}\right)Y_{s_u},Y_{t_1}\right)\right| = \left|Eg\left(Y_{s_1},\dots,Y_{s_u}\right)Y_{s_u}Y_{t_1}\right| \\
=\ & \frac{1}{n}\left|Eg\left(Y_{s_1},\dots,Y_{s_u}\right)\sum_{k',l'=1}^{d} c_{k'l'}\underbrace{\left(\sum_{i',j'=1}^{c}\theta_{k'i',s_u}\xi_{i',s_u}\theta_{l'j',s_u}\xi_{j',s_u}-\Sigma_{k'l',s_u}\right)}_{A_{k'l'}}\right.
\end{aligned}
$$

$$
\begin{aligned}
&\left.\cdot\sum_{k,l=1}^{d} c_{kl}\left(\sum_{i,j=1}^{c}\theta_{ki,t_1}\xi_{i,t_1}\theta_{lj,t_1}\xi_{j,t_1}-\Sigma_{kl,t_1}\right)\right| \\
=\ & \frac{1}{n}\Bigg|E\left(g\left(Y_{s_1},\ldots,Y_{s_u}\right)\sum_{k,l,k',l'=1}^{d}\sum_{i,j=1}^{c}c_{k'l'}A_{k'l'}c_{kl}\theta_{ki,t_1}\theta_{lj,t_1}\left(\xi_{i,t_1}\xi_{j,t_1}-\widetilde{\xi}_{i,t_1}^{(r)}\widetilde{\xi}_{j,t_1}^{(r)}\right)\right) \\
&\underbrace{+E\left(g\left(Y_{s_1},\ldots,Y_{s_u}\right)\sum_{k,l,k',l'=1}^{d}\sum_{i,j=1}^{c}c_{k'l'}A_{k'l'}c_{kl}\left(\sum_{i,j=1}^{c}\theta_{ki,t_1}\theta_{lj,t_1}\widetilde{\xi}_{i,t_1}^{(r)}\widetilde{\xi}_{j,t_1}^{(r)}-\Sigma_{kl,t_1}\right)\right)}_{\Psi}\Bigg| \\
=\ & \frac{1}{n}\Bigg|E\left(g\left(Y_{s_1},\ldots,Y_{s_u}\right)\sum_{k,l,k',l'=1}^{d}\sum_{i,j=1}^{c}c_{k'l'}A_{k'l'}c_{k,l}\theta_{ki,t_1}\theta_{lj,t_1}\left(\xi_{i,t_1}+\widetilde{\xi}_{i,t_1}^{(r)}\right)\left(\xi_{j,t_1}-\widetilde{\xi}_{j,t_1}^{(r)}\right)\right) \\
&\underbrace{+E\left(g\left(Y_{s_1},\ldots,Y_{s_u}\right)\sum_{k,l,k',l'=1}^{d}\sum_{i,j=1}^{c}c_{k'l'}A_{k'l'}c_{kl}\theta_{ki,t_1}\theta_{lj,t_1}\left(\widetilde{\xi}_{i,t_1}^{(r)}\xi_{j,t_1}-\xi_{i,t_1}\widetilde{\xi}_{j,t_1}^{(r)}\right)\right)}_{\Phi}+\Psi\Bigg|
\end{aligned}
$$

where the second inequality is due to the Jensen inequality, the last two inequality are due to the Cauchy-Schwarz inequality.
The independence between $\xi_{i,s}$ and $\widetilde{\xi}_{j,t_1}^{(r)}$ for all $s \leq s_u$ and all $i, j \in \{1, \ldots, c\}$ leads to the independence of $g\left(Y_{s_1}, \ldots, Y_{s_u}\right)$ and $\widetilde{\xi}_{i,t_1}^{(r)}\widetilde{\xi}_{j,t_1}^{(r)}$. Due to the computation (4.12), we have then

$$
\begin{aligned}
\Psi &= E\left(g\left(Y_{s_1},\ldots,Y_{s_u}\right)\sum_{k',l'=1}^{d}c_{k'l'}A_{k'l'}\right)\sum_{k,l=1}^{d}c_{kl}\left(\sum_{i,j=1}^{c}\theta_{ki,t_1}\theta_{lj,t_1}E\left(\widetilde{\xi}_{i,t_1}^{(r)}\widetilde{\xi}_{j,t_1}^{(r)}\right)-\Sigma_{kl,t_1}\right) \\
&= E\left(g\left(Y_{s_1},\ldots,Y_{s_u}\right)\sum_{k',l'=1}^{d}c_{k'l'}A_{k'l'}\right)\sum_{k,l=1}^{d}c_{kl}\left(\sum_{i=1}^{c}\theta_{ki,t_1}\theta_{li,t_1}-\Sigma_{kl,t_1}\right) \\
&= 0.
\end{aligned}
$$

Considering the Term Φ, if $i = j$, $\widetilde{\xi}_{i,t_1}^{(r)}\xi_{j,t_1} - \xi_{i,t_1}\widetilde{\xi}_{j,t_1}^{(r)} = 0$, if $i \neq j$, $\widetilde{\xi}_{i,t_1}^{(r)}$ and ξ_{j,t_1} are independent. Together with the independence between $g\left(Y_{s_1}, \ldots, Y_{s_u}\right)$ and $\widetilde{\xi}_{i,t_1}^{(r)}$, and $E\widetilde{\xi}_{i,t_1}^{(r)} = 0$, we have $\Phi = 0$.
Thereafter, we have for $\sup\limits_{k,l\in\{1,\ldots,d\}} c_{kl} < C_0$, that

$$
\begin{aligned}
&\left|Cov\left(g\left(Y_{s_1},\ldots,Y_{s_u}\right)Y_{s_u},Y_{t_1}\right)\right| \\
\leq\ & \frac{1}{n}C_0^2\|g\|_\infty\left|E\sum_{k,l,k',l'=1}^{d}\sum_{i,j=1}^{c}A_{k'l'}\theta_{ki,t_1}\theta_{lj,t_1}\left(\xi_{i,t_1}+\widetilde{\xi}_{i,t_1}^{(r)}\right)\left(\xi_{j,t_1}-\widetilde{\xi}_{j,t_1}^{(r)}\right)\right| \\
\leq\ & \frac{1}{n}C_0^2\|g\|_\infty\sum_{k,l,k',l'=1}^{d}\sum_{i,j=1}^{c}\theta_{ki,t_1}\theta_{lj,t_1}E\left|A_{k'l'}\left(\xi_{i,t_1}+\widetilde{\xi}_{i,t_1}^{(r)}\right)\left(\xi_{j,t_1}-\widetilde{\xi}_{j,t_1}^{(r)}\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{n} C_0^2 \|g\|_\infty \sum_{k,l,k',l'=1}^{d} \sum_{i,j=1}^{c} \theta_{ki,t_1}\theta_{lj,t_1} \sqrt{E\left(A_{k'l'}\left(\xi_{i,t_1}+\widetilde{\xi}_{i,t_1}^{(r)}\right)\right)^2} \sqrt{E\left(\xi_{j,t_1}-\widetilde{\xi}_{j,t_1}^{(r)}\right)^2} \\
&\leq \frac{1}{n} C_0^2 \|g\|_\infty \sum_{k,l,k',l'=1}^{d} \sum_{i,j=1}^{c} \theta_{ki,t_1}\theta_{lj,t_1} \sqrt[4]{E\left(A_{k'l'}\right)^4} \sqrt[4]{E\left(\xi_{i,t_1}+\widetilde{\xi}_{i,t_1}^{(r)}\right)^4} \sqrt{E\left(\xi_{j,t_1}-\widetilde{\xi}_{j,t_1}^{(r)}\right)^2}
\end{aligned}
$$

Since that the 8th order moment of $(\xi_{i,t})_{t\in\mathbb{Z}}$ for all $i \in \{1,\ldots,c\}$ exists, we have $E\left(A_{k'l'}\right)^4 \leq \infty$ and $E\left(\xi_{i,t_1}+\widetilde{\xi}_{i,t_1}^{(r)}\right)^4 \leq \infty$. Due to $\|g\|_\infty \leq 1$ and $GMC(2)$, we find then a summable sequence $\eta_{1,r}$, i.e.

$$
\eta_{1,r} := \frac{1}{n} C_0^2 \sum_{k,l,k',l'=1}^{d} \sum_{i,j=1}^{c} \theta_{ki,t_1}\theta_{lj,t_1} \sqrt[4]{E\left(A_{k'l'}\right)^4} \sqrt[4]{E\left(\xi_{i,t_1}+\widetilde{\xi}_{i,t_1}^{(r)}\right)^4} C\vartheta^r.
$$

such that

$$
\left|Cov\left(g\left(Y_{s_1},\ldots,Y_{s_u}\right)Y_{s_u},Y_{t_1}\right)\right| \leq n^{-1}\eta_{1,r}.
$$

Analogously, we can find a summable sequence $\eta_{2,r}$, so that

$$
\left|Cov\left(g\left(Y_{s_1},\ldots,Y_{s_u}\right),Y_{t_1}Y_{t_2}\right)\right| \leq n^{-1}\eta_{2,r}.
$$

Let $\eta_r = \max\{\eta_{1,r},\eta_{2,r}\}$. We have exactly the conditions of weak dependence needed for the central limit theorem for triangular arrays of possibly nonstationary random variables of Neumann (2013). This Theorem can be applied to obtain the asymptotic normality given by (4.13). With the Cramér-Wold Device, we obtain the following multivariate asymptotic normality:

$$
\sqrt{n}\left(\sum_{t=1}^{n} vec\left(\underline{X}_t\underline{X}_t'\right) - \sum_{t=1}^{n} vec\left(E\underline{X}_t\underline{X}_t'\right)\right) \longrightarrow \mathcal{N}\left(0,\underline{\widetilde{V}}\right).
$$

The assertion follows with the Lemma of Slutsky. □

Proofs for the local block bootstrap

Recall that a local windows has $2N+1$ data points. We use the shorthand $\theta\theta_{kluv,i+r} = \theta_{ku,i+r}\theta_{lv,i+r}$, $\xi\xi_{uv,i+r} = \xi_{u,i+r}\xi_{v,i+r}$, so that i.e. $X_{(k)i+r}X_{(l)i+r} = \sum_{u,v=1}^{c} \theta\theta_{kluv,i+r}\xi\xi_{uv,i+r}$, and $\overline{\xi\xi}_{uv,i+r} = \xi_{u,i+r}\xi_{v,i+r} - \frac{1}{2N+1}\sum_{p=-N}^{N}\xi_{u,i+p}\xi_{v,i+p}$ in the following proof.

Proof of Lemma 4.7
Assuming that $\left\lceil\frac{i}{B}\right\rceil - \left\lceil\frac{j}{B}\right\rceil = 0$, and neither i nor j are end points, we have

$$
Cov^*\left(X_{(k)i}^*X_{(l)i}^*, X_{(k')j}^*X_{(l')j}^*\right)
$$

$$
\begin{aligned}
&= \frac{1}{2N+1}\sum_{r=-N}^{N}\left(X_{(k)i+r}X_{(l)i+r} - \frac{1}{2N+1}\sum_{p=-N}^{N} X_{(k)i+p}X_{(l)i+p}\right) \\
&\quad \cdot \left(X_{(k')j+r}X_{(l')j+r} - \frac{1}{2N+1}\sum_{q=-nB}^{N} X_{(k')j+q}X_{(l')j+q}\right) \\
&= \frac{1}{n^2(2N+1)}\sum_{r=-N}^{N}\left(\sum_{u,v=1}^{c}\theta\theta_{kluv,i+r}\xi\xi_{uv,i+r} - \frac{1}{2N+1}\sum_{p=-N}^{N}\sum_{u'v'=1}^{c}\theta\theta_{klu'v',i+p}\xi\xi_{u'v',i+p}\right) \\
&\quad \cdot \left(\sum_{u,v=1}^{c}\theta\theta_{k'l'uv,j+r}\xi\xi_{uv,j+r} - \frac{1}{2N+1}\sum_{q=-N}^{N}\sum_{u'v'=1}^{c}\theta\theta_{k'l'u'v',j+q}\xi\xi_{u'v',j+q}\right) \\
&= \frac{1}{n^2(2N+1)}\sum_{r=-N}^{N}\left(\sum_{u,v=1}^{c}\theta\theta_{kluv,i+r}\overline{\xi\xi}_{uv,i+r}\right. \\
&\quad \left. + \frac{1}{2N+1}\sum_{p=-N}^{N}\sum_{u',v'=1}^{c}\left(\theta\theta_{klu'v',i+r} - \theta\theta_{klu'v',i+p}\right)\xi\xi_{u'v',i+p}\right) \\
&\quad \cdot \left(\sum_{u,v=1}^{c}\theta\theta_{k'l'uv,j+r}\overline{\xi\xi}_{uv,j+r} + \frac{1}{2N+1}\sum_{p=-N}^{N}\sum_{u',v'=1}^{c}\left(\theta\theta_{k'l'u'v',j+r} - \theta\theta_{k'l'u'v',j+p}\right)\xi\xi_{u'v',j+p}\right) \\
&= \frac{1}{n^2(2N+1)}\sum_{r=-N}^{N}\left(\sum_{u,v=1}^{c}\theta\theta_{kluv,i+r}\overline{\xi\xi}_{uv,i+r} + A_1\right)\left(\sum_{u,v=1}^{c}\theta\theta_{k'l'uv,j+r}\overline{\xi\xi}_{uv,j+r} + A_2\right) \\
&= \frac{1}{n^2(2N+1)}\sum_{r=-N}^{N}\left(\sum_{u,v,u',v'=1}^{c}\theta\theta_{kluv,i+r}\theta\theta_{k'l'u'v',j+r}\overline{\xi\xi}_{uv,i+r}\overline{\xi\xi}_{u'v',j+r}\right. \\
&\quad \left. + \sum_{u,v=1}^{c}\theta\theta_{kluv,i+r}\overline{\xi\xi}_{uv,i+r}A_2 + \sum_{u',v'=1}^{c}\theta\theta_{k'l'u'v',j+r}\overline{\xi\xi}_{u'v',j+r}A_1 + A_1A_2\right),
\end{aligned}
$$

where A_1 und A_2 are the obvious notations.
Recalling (4.5), it holds uniformly in $i \in \{1,\ldots,n\}$ that

$$
\sup_{r,p\in\{-N,\ldots,N\}} \left|\theta\theta_{klu'v',i+r} - \theta\theta_{klu'v',i+p}\right| = \mathcal{O}\left(\frac{N}{n}\right),
$$

and since that the eighth order moment of ξ_i is finite, it holds

$$
\begin{aligned}
&\sup_{r,p\in\{-N,\ldots,N\}}\left|\sum_{u,v=1}^{c}\theta\theta_{kluv,i+r}\overline{\xi\xi}_{uv,i+r}A_2\right| = \mathcal{O}_p\left(\frac{N}{n}\right) \\
&\sup_{r,p\in\{-N,\ldots,N\}}\left|\sum_{u',v'=1}^{c}\theta\theta_{k'l'u'v',j+r}\overline{\xi\xi}_{u'v',j+r}A_1\right| = \mathcal{O}_p\left(\frac{N}{n}\right) \\
&\sup_{r,p\in\{-N,\ldots,N\}}|A_1A_2| = \mathcal{O}_p\left(\frac{N^2}{n^2}\right)
\end{aligned}
$$

Further we have

$$
\begin{aligned}
& \frac{1}{n^2(2N+1)} \sum_{r=-N}^{N} \left(\sum_{u,v,u',v'=1}^{c} \theta\theta_{kluv,i+r} \theta\theta_{k'l'u'v',j+r} \overline{\xi\xi}_{uv,i+r} \overline{\xi\xi}_{u'v',j+r} \right) \\
= \; & \frac{1}{n^2(2N+1)} \sum_{r=-N}^{N} \left(\sum_{u,v,u',v'=1}^{c} \theta\theta_{kluv,i} \theta\theta_{k'l'u'v',j} \overline{\xi\xi}_{uv,i+r} \overline{\xi\xi}_{u'v',j+r} \right) \\
& + \frac{1}{n^2(2N+1)} \sum_{r=-N}^{N} \left(\sum_{u,v,u',v'=1}^{c} \left(\theta\theta_{kluv,i+r} \theta\theta_{k'l'u'v',j+r} - \theta\theta_{kluv,i} \theta\theta_{k'l'u'v',j} \right) \right. \\
& \left. \cdot \, \overline{\xi\xi}_{uv,i+r} \overline{\xi\xi}_{u'v',j+r} \right) \\
= \; & B1 + B2
\end{aligned}
$$

In the same way of (4.5), we can show

$$
\sup_{r \in \{-N,\dots,N\}} \left| \theta\theta_{kluv,i+r} \theta\theta_{k'l'u'v',j+r} - \theta\theta_{kluv,i} \theta\theta_{k'l'u'v',j} \right| = \mathcal{O}\left(\frac{N}{n} \right).
$$

This leads to $\sup_{r \in \{-N,\dots,N\}} |B2| = \mathcal{O}_p\left(\frac{N}{n^3} \right)$.

$$
\begin{aligned}
B1 \; = \; & \frac{1}{n^2} \sum_{u,v,u',v'=1}^{c} \left(\theta\theta_{kluv,i} \theta\theta_{k'l'u'v',j} \frac{1}{2N+1} \sum_{r=-N}^{N} \left(\overline{\xi\xi}_{uv,i+r} \overline{\xi\xi}_{u'v',j+r} \right) \right) \\
= \; & \frac{1}{n^2} \sum_{u,v,u',v'=1}^{c} \left(\theta\theta_{kluv,i} \theta\theta_{k'l'u'v',j} \left(Cov\left(\xi_{u,i}\xi_{v,i}, \xi_{u',j}\xi_{v',j} \right) + \mathcal{O}_p\left(N^{-\frac{1}{2}} \right) \right) \right) \\
= \; & Cov\left(\sum_{u,v,=1}^{c} \frac{1}{n} \theta_{ku,i}\xi_{u,i}\theta_{lv,i}\xi_{v,i}, \sum_{u',v',=1}^{c} \frac{1}{n} \theta_{k'u',j}\xi_{u',j}\theta_{l'v',j}\xi_{v',j} \right) + \mathcal{O}_p\left(\frac{1}{n^2\sqrt{N}} \right) \\
= \; & Cov\left(X_{(k)i}X_{(l)i}, X_{(k')j}X_{(l')j} \right) + \mathcal{O}_p\left(\frac{1}{n^2\sqrt{N}} \right).
\end{aligned}
$$

All together, we have

$$
Cov^*\left(X^*_{(k)i}X^*_{(l)i}, X^*_{(k')j}X^*_{(l')j} \right) = Cov\left(X_{(k)i}X_{(l)i}, X_{(k')j}X_{(l')j} \right) + \mathcal{O}_p\left(\frac{1}{n^2\sqrt{N}} \right) + \mathcal{O}_p\left(\frac{N}{n^3} \right)
$$

which is the desired result. □

Proof of Theorem 4.8

The asymptotic behavior of $\underline{T}_n^{LBB}$ from Theorem 4.8 will be considered in two steps. We estimate first the asymptotic covariance matrix of $\underline{T}_n^{LBB}$, and then prove the multivariate asymptotic normality by using the central limit theorem for the triangular array of the sum of independent non-identically distributed random variables.

Asymptotic covariance:
For the computation of the asymptotic covariance, we need to separate the bootstrapped data into some subgroups, in order to deal with the boundary effect. Recall that a local window has $2N+1$ data points, a block of the local block bootstrap has B data points and $b = \left\lceil \frac{n}{B} \right\rceil$ is the number of the blocks. Let $e_1 = \left\lceil \frac{N}{B} \right\rceil \in \mathbb{N}$ and $e_2 = \left\lceil \frac{n-N}{B} \right\rceil \in \mathbb{N}$, we separate the bootstrap sample in three subgroups: $\{\underline{X}_1^*, \ldots, \underline{X}_{e_1 B}^*\}$, $\{\underline{X}_{e_1 B+1}^*, \ldots, \underline{X}_{e_2 B}^*\}$ and $\{\underline{X}_{e_2 B+1}^*, \ldots, \underline{X}_n^*\}$, which consists of e_1, $e_2 - e_1$ and $b - e_2$ independent blocks respectively, and the three subgroups are independent. The first and the third subgroup could include blocks, which do not capture the autocovariance structure correctly because of Step 1 in the bootstrap procedure. We have for any $k, l \in \{1, \ldots, n\}$, that

$$\begin{aligned}\sum_{t=1}^{n} X_{(k)t}^* X_{(l)t}^* &= \sum_{s=0}^{e_1-1} \sum_{r=1}^{B} X_{(k)sB+r}^* X_{(l)sB+r}^* + \sum_{s=e_1}^{e_2-1} \sum_{r=1}^{B} X_{(k)sB+r}^* X_{(l)sB+r}^* \\ &\quad + \sum_{t=e_2 B+1}^{n} X_{(k)t}^* X_{(l)t}^* = A1_{klt} + A2_{klt} + A3_{klt}\end{aligned}$$

The elements of the asymptotic covariance matrix of $\underline{T}_n^{LBB}$ is computed as follows

$$\begin{aligned}& V_{klk'l',n}^{LBB} \\ := \;& nCov^*\left(\sum_{t=1}^{n} \left(X_{(k)t}^* X_{(l)t}^* - E^* X_{(k)t}^* X_{(l)t}^* \right), \sum_{t'=1}^{n} \left(X_{(k')t'}^* X_{(l')t'}^* - E^* X_{(k')t'}^* X_{(l')t'}^* \right) \right) \\ = \;& nCov^*\left(A1_{klt} + A2_{klt} + A3_{klt}, A1_{k'l't'} + A2_{k'l't'} + A3_{k'l't'} \right) \\ = \;& nCov^*\left(A1_{klt}, A1_{k'l't'} \right) + nCov^*\left(A2_{klt}, A2_{k'l't'} \right) + nCov^*\left(A3_{klt}, A3_{k'l't'} \right).\end{aligned}$$

Because of the independence between the blocks, we have

$$\begin{aligned}& nCov^*\left(A2_{klt}, A2_{k'l't'} \right) \\ = \;& n \sum_{s=e_1}^{e_2} \left(Cov^*\left(\sum_{r=1}^{B} X_{(k)sB+r}^* X_{(l)sB+r}^*, \sum_{r'=1}^{B} X_{(k')sB+r'}^* X_{(l')sB+r'}^* \right) \right) \\ = \;& n \sum_{s=e_1}^{e_2} \left(\sum_{r,r'=1}^{B} Cov^*\left(X_{(k)sB+r}^* X_{(l)sB+r}^*, X_{(k')sB+r'}^* X_{(l')sB+r'}^* \right) \right)\end{aligned}$$

Due to Lemma 4.7 and the computation of the asymptotic covariance in the proof of Theorem 4.5 , we obtain

$$
\begin{aligned}
& nCov^*\left(A2_{klt}, A2_{k'l't'}\right) \\
= \;& n \sum_{s=e_1}^{e_2} \sum_{r,r'=1}^{B} \Big(Cov\left(X_{(k)sB+r} X_{(l)sB+r}, X_{(k')sB+r'} X_{(l')sB+r'} \right) \\
& + \mathcal{O}_p\left(\frac{N}{n^3}\right) + \mathcal{O}_p\left(\frac{1}{n^2\sqrt{N}}\right)\Big) \\
= \;& \frac{1}{n} \sum_{s=e_1}^{e_2} \sum_{r,r'=1}^{B} \left(\sum_{u=1}^{c} \theta\theta_{kluu,sB+r} \theta\theta_{k'l'uu,sB+r'} \left(\gamma_{\xi_1^2}(r-r') - 2\gamma_{\xi_1}^2(r-r') \right) \right. \\
& \left. + \left(\Sigma_{kk',sB+r}\Sigma_{ll',sB+r'} + \Sigma_{kl',sB+r}\Sigma_{k'l,sB+r'} \right) \gamma_{\xi_1}^2(r-r') \right) \\
& + \mathcal{O}_p\left(\frac{BN}{n}\right) + \mathcal{O}_p\left(\frac{B}{\sqrt{N}}\right)
\end{aligned}
$$

Since $\sup_{r,r' \in \{1,\ldots,B\}} \left|\theta\theta_{kluv,sB+r}\theta\theta_{k'l'u'v',sB+r'} - \theta\theta_{kluv,sB}\theta\theta_{k'l'u'v',sB}\right| = \mathcal{O}\left(\frac{B}{n}\right)$, we have

$$
\begin{aligned}
& nCov^*\left(A2_{klt}, A2_{k'l't'}\right) \\
= \;& \frac{1}{n} \sum_{s=e_1}^{e_2} \sum_{r,r'=1}^{B} \left(\sum_{u=1}^{c} \theta\theta_{kluu,sB} \theta\theta_{k'l'uu,sB} \left(\gamma_{\xi_1^2}(r-r') - 2\gamma_{\xi_1}^2(r-r') \right) \right. \\
& \left. + \left(\Sigma_{kk',sB}\Sigma_{ll',sB} + \Sigma_{kl',sB}\Sigma_{k'l,sB} \right) \gamma_{\xi_1}^2(r-r') \right) + \mathcal{O}_p\left(\frac{BN}{n}\right) + \mathcal{O}_p\left(\frac{B}{\sqrt{N}}\right) \\
& + \frac{1}{n} \sum_{s=e_1}^{e_2} \sum_{r,r'=1}^{B} \left(\mathcal{O}\left(\frac{B}{n}\right) \left(\gamma_{\xi_1^2}(r-r') - 2\gamma_{\xi_1}^2(r-r') \right) + \mathcal{O}\left(\frac{B}{n}\right) \gamma_{\xi_1}^2(r-r') \right) \\
= \;& D1 + \mathcal{O}_p\left(\frac{BN}{n}\right) + \mathcal{O}_p\left(\frac{B}{\sqrt{N}}\right) + D2.
\end{aligned}
$$

Considering the number of blocks in the subgroup $\{\underline{X}^*_{e_1B+1}, \ldots, \underline{X}^*_{e_2B}\}$, we have

$$
\frac{e_2 - e_1 + 1}{n/B} = \frac{B\left(\lceil \frac{n-N}{B} \rceil - \lceil \frac{N}{B} \rceil\right)}{n} \overset{n\to\infty}{\longrightarrow} 1.
$$

Due to Lemma 4.10 and Riemann sum, it holds, as $n \to \infty$, that

$$
\begin{aligned}
D1 \;=\; & \frac{B}{n} \sum_{s=e_1}^{e_2} \sum_{u=1}^{c} \theta\theta_{kluu,sB} \theta\theta_{k'l'uu,sB} \frac{1}{B} \sum_{r,r'=1}^{B} \left(\gamma_{\xi_1^2}(r-r') - 2\gamma_{\xi_1}^2(r-r') \right) \\
& + \frac{B}{n} \sum_{s=e_1}^{e_2} \left(\Sigma_{kk',sB}\Sigma_{ll',sB} + \Sigma_{kl',sB}\Sigma_{k'l,sB} \right) \frac{1}{B} \sum_{r,r'=1}^{B} \gamma_{\xi_1}^2(r-r') \\
\longrightarrow \; & \int_0^1 \left[\sum_{u=1}^{c} \theta_{ki}(u)\theta_{li}(u)\theta_{k'i}(u)\theta_{l'i}(u) \left(\sum_{h=-\infty}^{\infty} \gamma_{\xi_1^2}(h) - 2 \sum_{h=-\infty}^{\infty} \gamma_{\xi_1}^2(h) \right) \right.
\end{aligned}
$$

$$+\Sigma_{kk'}(u)\Sigma_{ll'}(u)\sum_{h=-\infty}^{\infty}\gamma_{\xi_1}^2(h)+\Sigma_{kl'}(u)\Sigma_{k'l}(u)\sum_{h=-\infty}^{\infty}\gamma_{\xi_1}^2(h)\Bigg]du=\widetilde{V}_{klk'l'}.$$

Duo to the Kronecker-Lemma $D2 \to 0$, as $n \to \infty$. We have then

$$nCov^*\left(A2_{klt}, A2_{k'l't'}\right) \xrightarrow{p} \widetilde{V}_{klk'l'}$$

In the same way as before and because of Step 1 of the bootstrap procedure, we have

$$\begin{aligned}
& nCov^*\left(A1_{klt}, A1_{k'l't'}\right) \\
= \; & n\sum_{s=0}^{e_1-1}\left(Cov^*\left(\sum_{r,r'=1}^{B}X^*_{(k)sB+r}X^*_{(l)sB+r},\sum_{r=1}^{B}X^*_{(k')sB+r'}X^*_{(l')sB+r'}\right)\right) \\
\leq \; & \frac{2B}{n}\sum_{s=0}^{e_1-1}\sum_{u=1}^{c}\theta\theta_{kluu,sB}\theta\theta_{k'l'uu,sB}\frac{1}{B}\sum_{r,r'=1}^{B}\left(\gamma_{\xi_1^2}(r-r')-2\gamma^2_{\xi_1}(r-r')\right) \\
& +\frac{B}{n}\sum_{s=0}^{e_1-1}\left(\Sigma_{kk',sB}\Sigma_{ll',sB}+\Sigma_{kl',sB}\Sigma_{k'l,sB}\right)\frac{1}{B}\sum_{r,r'=1}^{B}\gamma^2_{\xi_1}(r-r')+o_p(1)
\end{aligned}$$

We have $\dfrac{e_1}{n/B}=\dfrac{B\lceil\frac{N}{B}\rceil}{n}\to 0$, $\sum_{h=-\infty}^{\infty}\gamma_{\xi_1}(h)<\infty$ and $\sum_{h=-\infty}^{\infty}\gamma_{\xi_1^2}(h)<\infty$, which lead to

$$nCov^*\left(A1_{klt}, A1_{k'l't'}\right) \xrightarrow{p} 0$$

and analogously

$$nCov^*\left(A3_{klt}, A3_{k'l't'}\right) \xrightarrow{p} 0.$$

Together we have the asymptotic bootstrapped covariance matrix with the elements

$$V^{LBB}_{klk'l',n} \xrightarrow{p} \widetilde{V}_{klk'l'}. \tag{4.14}$$

Asymptotic normality:

Analogously to the proof of Theorem 4.5, we need to show that for any real constants $\{c_{kl} : k,l \in \{1,\ldots,d\}\}$, it holds, as $n \to \infty$,

$$\underline{C}'\underline{T}_n^{LBB}=\sqrt{n}\sum_{t=1}^{n}\underline{C}'\left(vec\left(\underline{X}_t^*\underline{X}_t^{*\prime}\right)-vec\left(\underline{X}_t\underline{X}_t'\right)\right)\xrightarrow{d}\mathcal{N}\left(0,\underline{C}'\widetilde{\underline{V}}\underline{C}\right), \tag{4.15}$$

where $\underline{C} := vec\left(\left[c_{kl} : k,l \in \{1,\ldots,d\}\right]_{d\times d}\right)$.

(4.14) leads to, as $n \to \infty$, that

$$Var\left(\underline{C}'\underline{T}_n^{LBB}\right) \xrightarrow{p} \underline{C}'\widetilde{\underline{V}}\underline{C}.$$

Now let

$$\begin{aligned} Y_s &:= \sum_{r=1}^{B} \sqrt{n}\underline{C}' \left(vec\left(\underline{X}^*_{sB+r} \underline{X}^{*'}_{sB+r} \right) - vec\left(\underline{X}_{sB+r} \underline{X}'_{sB+r} \right) \right) \\ &= \sum_{r=1}^{B} \sqrt{n} \sum_{k,l=1}^{d} c_{kl} \left(X^*_{(k)sB+r} X^*_{(l)sB+r} - X_{(k)sB+r} X_{(l)sB+r} \right), \end{aligned}$$

$s \in \{0, 1, \ldots, b-1\}$, if $b = \frac{n}{B}$. If $b > \frac{n}{B}$, $(Y_s)_{s=0,1,\ldots,b-2}$ will be defined as before, but

$$Y_{b-1} := \sum_{t=B(b-1)}^{n} \sqrt{n}\underline{C}' \left(vec\left(\underline{X}^*_t \underline{X}^{*'}_t \right) - vec\left(\underline{X}_t \underline{X}'_t \right) \right).$$

$\underline{C}'\underline{T}_n^{LBB}$ is the sum of the independent non-identically distributed variables $(Y_s)_{s=0,\ldots,b-1}$. We need to show that Y_s's satisfy the Lindeberg condition. The central limit theorem for the triangular array of the sum of independent, non-identically distributed random variables yields the asymptotic normality given by (4.15).

Lindeberg condition:
For the convenience of presentation, we assume that $n = bB$. Let $\zeta > 0$, $\delta = \Delta/8$ and $X^*_{(k)t} = \sum_{i=1}^{c} \frac{1}{\sqrt{n}} \theta^*_{ki,t} \varepsilon^*_{i,t}$, $k \in \{1, \ldots, d\}$ and $t \in \{1, \ldots, n\}$. Using the shorthand $\theta\theta_{klij,sB+r} = \theta_{ki,sB+r}\theta_{lj,sB+r}$, $\xi\xi_{ij,sB+r} = \xi_{i,sB+r}\xi_{j,sB+r}$ and the shorthand in the same way for the bootstrapped values, we have

$$\begin{aligned} & \sum_{s=0}^{b-1} E^* \left[Y_s^2 \mathbb{1}_{\{|Y_s| > \zeta\}} \right] \\ \leq\ & \sum_{s=0}^{b-1} \left(E^* \left[Y_s^{2(1+\delta)} \right] \right)^{\frac{1}{1+\delta}} \left(E^* \left[\mathbb{1}_{\{|Y_s| > \zeta\}} \right] \right)^{\frac{\delta}{1+\delta}} \\ =\ & \frac{1}{b} \sum_{s=0}^{b-1} \left(E^* \left[\frac{1}{B} \sum_{r=1}^{B} \sum_{k,l=1}^{d} \sum_{i,j=1}^{c} c_{kl} \left(\theta^*\theta^*_{klij,sB+r} \xi^*\xi^*_{ij,sB+r} - \theta\theta_{klij,sB+r} \xi\xi_{ij,sB+r} \right) \right]^{2(1+\delta)} \right)^{\frac{1}{1+\delta}} \\ & \cdot \left(E^* \left[\mathbb{1}_{\left\{ \left| \frac{1}{B} \sum_{r=1}^{B} \sum_{k,l=1}^{d} \sum_{i,j=1}^{c} c_{kl} \left(\theta^*\theta^*_{klij,sB+r} \xi^*\xi^*_{ij,sB+r} - \theta\theta_{klij,sB+r} \xi\xi_{ij,sB+r} \right) \right| > \zeta\sqrt{b} \right\}} \right] \right)^{\frac{\delta}{1+\delta}} \\ \leq\ & \frac{1}{b} \sum_{s=0}^{b-1} \left(E^* \left[\frac{1}{B} \sum_{r=1}^{B} \sum_{k,l=1}^{d} \sum_{i,j=1}^{c} c_{kl} \left(\theta^*\theta^*_{klij,sB+r} \xi^*\xi^*_{ij,sB+r} - \theta\theta_{klij,sB+r} \xi\xi_{ij,sB+r} \right) \right]^{2(1+\delta)} \right)^{\frac{1}{1+\delta}} \\ & \cdot \left(E^* \left| \frac{1}{B} \sum_{r=1}^{B} \sum_{k,l=1}^{d} \sum_{i,j=1}^{c} c_{kl} \left(\theta^*\theta^*_{klij,sB+r} \xi^*\xi^*_{ij,sB+r} - \theta\theta_{klij,sB+r} \xi\xi_{ij,sB+r} \right) \right|^{2(1+\delta)} \right)^{\frac{\delta}{1+\delta}} \frac{1}{\zeta^{2\delta} b^{\delta}} \end{aligned}$$

$$\leq \frac{1}{\zeta^{2\delta} b^{1+\delta}} \sum_{s=0}^{b-1} E^* \left| \frac{1}{B} \sum_{r=1}^{B} \sum_{k,l=1}^{d} \sum_{i,j=1}^{c} c_{kl} \left(\theta^* \theta^*_{klij,sB+r} \xi^* \xi^*_{ij,sB+r} - \theta \theta_{klij,sB+r} \xi \xi_{ij,sB+r} \right) \right|^{2(1+\delta)}$$

where the first inequality is due to the Hölder inequality and the second one is due to the Markov inequality.

Since $E\left|\xi\right|_{i,t}^{8+\Delta} < \infty$ for all $i \in \{1, \ldots, c\}$ and $t \in \mathbb{Z}$, we can show, in the same way as in the proof of Lemma 4.7, that

$$E^* \left(\theta^* \theta^*_{klij,sB+r} \xi^* \xi^*_{ij,sB+r} \right)^{2(1+\delta)} = \theta \theta_{klij,sB+r} E \left(\xi \xi_{ij,sB+r} \right)^{2(1+\delta)} + o_p(1),$$

uniformly in $s \in \{0, \ldots, b-1\}$ and in $r \in \{1, \ldots, B\}$. Together with all $\theta_{ki,t} < \infty$, we have

$$E^* \left[\frac{1}{B} \sum_{r=1}^{B} \sum_{k,l=1}^{d} \sum_{i,j=1}^{c} c_{kl} \left(\theta^* \theta^*_{klij,sB+r} \xi^* \xi^*_{ij,sB+r} - \theta \theta_{klij,sB+r} \xi \xi_{ij,sB+r} \right) \right]^{2(1+\delta)} < \infty$$

in probability.

The Lindeberg condition is fulfilled. With the Cramér-Wold Device, we obtain the desired result. □

Proof for the local dependent wild bootstrap

Proof of Theorem 4.9

Analogously to the proof of the local block bootstrap, we consider the asymptotic behavior of $\underline{T}_n^{LDW}$ from Theorem 4.9 in two steps. The asymptotic covariance matrix of $\underline{T}_n^{LDW}$ will be estimated in the first step, and then the multivariate asymptotic normality will be proved by using he central limit theorem for the triangular array of the sum of independent non-identically distributed random variables.

Asymptotic covariance:

Recall that $Y_{kl,i} = X_{(k)i} X_{(l)i}$, $\overline{Y}_{kl,s} = \frac{1}{H} \sum_{i=1}^{H} Y_{kl,sH+i}$,

$Y^*_{kl,sH+i} = \overline{Y}_{kl,s} + \left(Y_{kl,sH+i} - \overline{Y}_{kl,s} \right) W_{sH+i}$, $s = 0, \ldots, m-1$, $i, j = 1, \ldots, H$.

We have

$$\begin{aligned} & Cov^* \left(Y^*_{kl,sH+i}, Y^*_{k'l',sH+j} \right) \\ = \; & Cov^* \left(\overline{Y}_{kl,s} + \left(Y_{kl,sH+i} - \overline{Y}_{kl,s} \right) W_{sH+i}, \overline{Y}_{k'l',s} + \left(Y_{k'l',sH+j} - \overline{Y}_{k'l',s} \right) W_{sH+j} \right) \end{aligned}$$

$$
\begin{aligned}
&= \left(Y_{kl,sH+i} - \overline{Y}_{kl,s}\right)\left(Y_{k'l',sH+j} - \overline{Y}_{k'l',s}\right) Cov\left(W_{sH+i}, W_{sH+j}\right) \\
&= \left(Y_{kl,sH+i} - \overline{Y}_{kl,s}\right)\left(Y_{k'l',sH+j} - \overline{Y}_{k'l',s}\right) K\left(\frac{i-j}{l}\right).
\end{aligned}
$$

Since $\sup_{i\in\{1,\ldots,H\}} |\theta\theta_{klpq,sH+i} - \theta\theta_{klpq,sH}| = \mathcal{O}\left(\frac{H}{n}\right)$ and the fourth order moment of ξ is finite, we have

$$
\begin{aligned}
Y_{kl,sH+i} - \overline{Y}_{kl,s} &= \frac{1}{n}\left(\sum_{p,q=1}^{c} \theta\theta_{klpq,sH+i}\xi\xi_{pq,sH+i} - \frac{1}{H}\sum_{i'=1}^{H}\sum_{p',q'=1}^{c} \theta\theta_{klp'q',sH+i'}\xi\xi_{p'q',sH+i'}\right) \\
&= \frac{1}{n}\left(\sum_{p,q=1}^{c} \theta\theta_{klpq,sH}\left(\xi\xi_{pq,sH+i} - \frac{1}{H}\sum_{i'=1}^{H}\xi\xi_{pq,sH+i'}\right)\right. \\
&\quad \left. + \sum_{p,q=1}^{c} \mathcal{O}\left(\frac{H}{n}\right)\left(\xi\xi_{pq,sH+i} - \frac{1}{H}\sum_{i'=1}^{H}\xi\xi_{pq,sH+i'}\right)\right) \\
&= \frac{1}{n}\left(\sum_{p,q=1}^{c} \theta\theta_{klpq,sH}\left(\xi\xi_{pq,sH+i} - \frac{1}{H}\sum_{i'=1}^{H}\xi\xi_{pq,sH+i'}\right) + \mathcal{O}_p\left(\frac{H}{n}\right)\right).
\end{aligned}
$$

In the second equality above, $\theta\theta_{klpq,sH+i}$ is replaced with by $\theta\theta_{klpq,sH}$.
Let $\overline{\xi\xi}_{pq,SH+i} := \xi\xi_{pq,sH+i} - \frac{1}{H}\sum_{i'=1}^{H}\xi\xi_{pq,sH+i'}$, then

$$
\begin{aligned}
&Cov^*\left(Y^*_{kl,sH+i}, Y^*_{k'l',sH+j}\right) \\
&= \frac{1}{n^2}\left(\sum_{p,q=1}^{c}\theta\theta_{klpq,sH}\overline{\xi\xi}_{pq,sH+i} + \mathcal{O}_p\left(\frac{H}{n}\right)\right) \\
&\quad \cdot \left(\sum_{p',q'=1}^{c}\theta\theta_{k'l'p'q',sH}\overline{\xi\xi}_{p'q',sH+j} + \mathcal{O}_p\left(\frac{H}{n}\right)\right) K\left(\frac{i-j}{l}\right) \\
&= \frac{1}{n^2}\sum_{p,q,p',q'=1}^{c}\theta\theta_{klpq,sH}\theta\theta_{k'l'p'q',sH}\overline{\xi\xi}_{pq,sH+i}\overline{\xi\xi}_{p'q',sH+j} K\left(\frac{i-j}{l}\right) + \mathcal{O}_p\left(\frac{H}{n^3}\right).
\end{aligned}
$$

The elements of the covariance matrix is computed in the following:

$$
\begin{aligned}
&V^{LDW}_{klk'l',n} \\
&:= nCov^*\left(\sum_{t=1}^{n}\left(Y^*_{kl,t} - E^*Y^*_{kl,t}\right), \sum_{t'=1}^{n}\left(Y^*_{k'l',t'} - E^*Y^*_{k'l',t'}\right)\right) \\
&= n\sum_{s=0}^{m-1}\sum_{i,j=1}^{H} Cov^*\left(Y^*_{kl,sH+i}, Y^*_{k'l',sH+j}\right) \\
&= \frac{1}{n}\sum_{s=0}^{m-1}\sum_{i,j=1}^{H}\sum_{p,q,p',q'=1}^{c}\theta\theta_{klpq,sH}\theta\theta_{k'l'p'q',sH}\overline{\xi\xi}_{pq,sH+i}\overline{\xi\xi}_{p'q',sH+j} K\left(\frac{i-j}{l}\right) + \mathcal{O}_p\left(\frac{H^2}{n}\right)
\end{aligned}
$$

According to Proposition 2.1 of Shao (2010), we have for $p = 1, \ldots, c$

$$\frac{1}{H}\sum_{i,j=1}^{H} \overline{\xi\xi}_{pp,sH+i}\overline{\xi\xi}_{pp,sH+j} K\left(\frac{i-j}{l}\right) = \sum_{h=-\infty}^{\infty} \gamma_{\xi_1^2}(h) + o_p\left(1\right),$$

and for $p, q = 1, \ldots, c$, $p \neq q$

$$\frac{1}{H}\sum_{i,j=1}^{H} \overline{\xi\xi}_{pq,sH+i}\overline{\xi\xi}_{pq,sH+j} K\left(\frac{i-j}{l}\right) = \sum_{h=-\infty}^{\infty} \gamma_{\xi_1}^2(h) + o_p\left(1\right).$$

Then we have, as $n \to \infty$, that

$$\begin{aligned}
& V_{klk'l'}^{LDW} && (4.16)\\
= \; & \frac{1}{m}\sum_{s=0}^{m-1}\left(\sum_{p=1}^{c} \theta\theta_{klpp,sH}\theta\theta_{k'l'pp,sH}\left(\sum_{h=-\infty}^{\infty}\gamma_{\xi_1^2}(h) + o_p\left(1\right)\right)\right)\\
& + \frac{1}{m}\sum_{s=0}^{m-1}\left(\sum_{p,q=1,p\neq q}^{c} \theta\theta_{klpq,sH}\theta\theta_{k'l'pq,sH}\left(\sum_{h=-\infty}^{\infty}\gamma_{\xi_1}^2(h) + o_p\left(1\right)\right)\right)\\
& + \frac{1}{m}\sum_{s=0}^{m-1}\left(\sum_{p,q=1,p\neq q}^{c} \theta\theta_{klpq,sH}\theta\theta_{k'l'qp,sH}\left(\sum_{h=-\infty}^{\infty}\gamma_{\xi_1}^2(h) + o_p\left(1\right)\right)\right) + \mathcal{O}_p\left(\frac{H^2}{n}\right)\\
\xrightarrow{p} \; & \int_0^1 \Bigg[\sum_{p=1}^{c}\theta_{kp}(u)\theta_{lp}(u)\theta_{k'p}(u)\theta_{l'p}(u)\sum_{h=-\infty}^{\infty}\gamma_{\xi_1^2}(h)\\
& + \sum_{p,q=1,p\neq q}^{c}\left(\theta_{kp}(u)\theta_{lq}(u)\theta_{k'p}(u)\theta_{l'q}(u) + \theta_{kp}(u)\theta_{lp}(u)\theta_{k'q}(u)\theta_{l'p}(u)\right)\sum_{h=-\infty}^{\infty}\gamma_{\xi_1}^2(h)\Bigg] du\\
= \; & \int_0^1 \Bigg[\sum_{p=1}^{c}\theta_{kp}(u)\theta_{lp}(u)\theta_{k'p}(u)\theta_{l'p}(u)\left(\sum_{h=-\infty}^{\infty}\gamma_{\xi_1^2}(h) - 2\sum_{h=-\infty}^{\infty}\gamma_{\xi_1}^2(h)\right)\\
& + \left(\Sigma_{kk'}(u)\Sigma_{ll'}(u) + \Sigma_{kl'}(u)\Sigma_{k'l}(u)\right)\sum_{h=-\infty}^{\infty}\gamma_{\xi_1}^2(h)\Bigg] du && (4.17)
\end{aligned}$$

which is exactly the element of the asymptotic covariance matrix $\underline{\widetilde{V}}$ given in Theorem 4.5.

Asymptotic normality:
Again, by the Cramér-Wold Device, it suffices to show that for any real constants $\{c_{kl} : k, l \in \{1, \ldots, d\}\}$, it holds, as $n \to \infty$, that

$$\underline{C}'\underline{T}_n^{LWD} = \sqrt{n}\sum_{t=1}^{n}\underline{C}'\left(vecr\left(\underline{Y}_t^*\right) - vec\left(\underline{X}_t\underline{X}_t'\right)\right) \xrightarrow{d} \mathcal{N}\left(0, \underline{C}'\underline{\widetilde{V}}\underline{C}\right),$$

where $\underline{C} := vec\Big([c_{kl} : k, l \in \{1, \ldots, d\}]_{d\times d}\Big)$.
(4.17) leads to, as $n \to \infty$, that

$$Var\left(\underline{C}'\underline{T}_n^{LDW}\right) \xrightarrow{p} \underline{C}'\widetilde{V}\underline{C}.$$

For $s \in \{0, 1, \ldots, m-1\}$, let

$$\underline{Z}_s := \sum_{r=1}^{H} \sqrt{n}\underline{C}'\left(vec\left(\overline{Y}^*_{sH+r}\right) - vec\left(\underline{X}_{sH+r}\underline{X}'_{sH+r}\right)\right),$$

In view of the independence of the local windows of the bootstrapped sample, we have the sum of independent non-identically distributed variables $(\underline{Z}_s)_{s=0,\ldots,m-1}$.

Lindeberg condition:
Let $\zeta > 0$, $\delta = \Delta/8$. It is sufficient to show, as $n \to \infty$, that

$$\sum_{s=0}^{m-1} E^*\left[Z_s^2 \mathbb{1}_{\{|Z_s|>\zeta\}}\right] \leq \sum_{s=0}^{m-1}\left(E^*\left[Z_s^{2(1+\delta)}\right]\right)^{\frac{1}{1+\delta}}\left(E^*\left[\mathbb{1}_{\{|Z_s|>\zeta\}}\right]\right)^{\frac{\delta}{1+\delta}} \xrightarrow{p} 0.$$

The further proof is quite similar to the one by the local block bootstrap. What we need are the conditions $E\left|\xi_t\right|^{8(1+\delta)} < \infty$ and $E\left|W_t\right|^{2(1+\delta)} < \infty$, which are given in the assumptions. These assumptions guarantee the summands are all of order $\mathcal{O}\left(\frac{1}{m^{1+\delta}}\right)$.
The Central limit theorem for the triangular array of the sum of independent, non-identically distributed random variables yields the asymptotic normality given by Theorem 4.9. □

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